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Biased extensive measurement: The general case

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Abstract

We develop a theory of biased extensive measurement which allows us to prove the existence of a ratio-scale without transitivity of indifference and with a property of homothetic invariance weaker than independence. These representations, which cover the cases of interval orders and of semiorders, reveal a unique biasing function smaller or equal to 1 that distorts extensive measurement and explains departures from its standard axioms. We interpret this biasing function as characterizing the qualitative influence of the underlying measurement process and we show that it induces a proportional indifference threshold.
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1. Introduction

This paper shows that extensive measurement is possible even when a distortion of the measurement process causes a lack of discrimination (intransitive indifference) and a lack of consistency (violation of independence). Moreover, such a distortion can be characterized with a unique biasing function. In this manner, and in conditions more general than theories of extensive measurement, our theory of biased extensive measurement provides both a measurement of objects and a measurement of the measuring process.

Theories of extensive measurement are important for the mathematical foundations of science because they establish the conditions for a fully quantitative measurement of attributes such as, for instance, mass, length or time duration. Generally speaking, they can be formulated as axioms about a non-empty ordering $\succ$ on a set A (of objects $x, y, z \ldots \in A$) and a binary (commutative, associative) operation $\circ$ on A that permit the construction of a ratio-scale $\varphi : A \rightarrow \mathbb{R}_{>0}$ verifying

$$x \succ y \iff \varphi(x) > \varphi(y), \tag{i}$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y). \tag{ii}$$

Property (i) reflects that the scale φ associates a number with each object in such a way that the empirical relation $\succ$ among objects is *represented* by the numerical relation $>$. Property (ii) reflects the often taken-for-granted additive property that "the value of x and y combined together equals the value of x plus the value of y". When property (ii) is verified, we say that the scale is *additive* and that the measurement is *extensive*. Another critical aspect of extensive measurement is that φ is a *ratio-scale*. This means that, if another scale φ' also verifies properties (i) and (ii), then there exists a positive number λ such that $\varphi' = \lambda\varphi$. We say that φ is unique up to a positive scaling transformation or up to multiplication by a positive scalar. This *uniqueness condition* is important because intuitive statements such that "the value of x is twice the value of y" and "the value of x minus the value of y is greater than the value of z minus the value of t" are *meaningful*: if such statements are true for a scale φ verifying properties (i) and (ii), then they are also true for *any other scale* φ' verifying

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properties (i) and (ii). More generally, we say that a ratio-scale *preserves the comparison of ratios and of differences*. Finally, ratio-scales are also important because they allow for all types of statistical measures (mean, standard deviation, coefficient of variation, etc.) and always imply an absolute zero (see Stevens, 1946 for a seminal discussion on the theory of scales of measurement. See Krantz, Luce, Suppes, & Tversky, 1971, Chapter 3 and Roberts, 1979, Section 3.2 for the foundations of extensive measurement).

Two groups of axioms are crucial to these theories. Firstly, the ordering must be *asymmetric*: $x \succ y \Rightarrow y \nsucc x$, and *negatively transitive:* $(x \nsucc y \text{ and } y \nsucc z) \Rightarrow x \nsucc z$. These properties imply that the ordering is *transitive*: $(x \succ y \text{ and } y \succ z) \Rightarrow x \succ z$ and that the associated indifference relation $\sim$ defined by $x \sim y \Leftrightarrow (x \nsucc y \text{ and } y \nsucc x)$ is also transitive. Secondly, the combination of the ordering and the operation must verify a form of consistency called monotonicity, translation-invariance, or *independence*: $x \succ y \Leftrightarrow (x \circ z \succ y \circ z \text{ for all } z \in A)$. Both transitivity of indifference and independence are *necessary conditions* for showing the existence of a ratio-scale verifying (i) and (ii) above (because these conditions hold for the triple $\langle \mathbb{R}, >, + \rangle$). When the measurement process is distorted or biased, however, indifference may not be transitive. For instance, we may not be able to distinguish between object x and object $x \circ y$ when y is sufficiently "small". Indeed, it has long been recognized that practical measurement is always faced by some lack of discrimination. Second, we may not observe that $x \succ y$ whenever $x \circ z \succ y \circ z$, reflecting some lack of consistency. A typical example in the social sciences is the decreasing marginal value for money. According to the theories of extensive measurement presented above, the existence of an additive ratio-scale cannot be proven when we observe a violation of transitive indifference or of independence. This may haveconsiderably restricted the applications of extensive measurement, in particular in the psychological sciences.

With our theory of biased extensive measurement, we aim to generalize the conditions under which extensive measurement is possible. These theories can be formulated as a collection of axioms about a non-empty ordering $\succ$ and a binary (commutative, associative) operation $\circ$ on A that permit the construction of a ratio-scale $\varphi : A \to \mathbb{R}_{>0}$ and a unique function $\sigma(x,y) : A \times A \to \mathbb{R}$ verifying

$$x \succ y \iff \sigma(x,y)\varphi(x) > \varphi(y), \quad (\text{i}')$$

$$\varphi(mx) = m\varphi(x), \quad (\text{ii}')$$

$$\sigma(mx,my) = \sigma(x,y), \quad (\text{iii}')$$

where $mx = x \circ \cdots \circ x$ (m times, $m \in \mathbb{N}_{>0}$). The scale φ provides a ratio-scale measure of each object or stimuli and increases linearly with the quantity of a given object (property (ii′)). The "biasing function" σ provides a measure of the distortion or bias of the measurement process and remains constant when the quantity of the objects increases (property (iii′)). A necessary condition for the existence of a scale φ verifying properties (i′), (ii′) and (iii′) is the property of *scale-invariance* or *homotheticity* $x \succ y \Leftrightarrow mx \succ my$ *for all* $m \in \mathbb{N}_{>0}$. Indeed, this condition will be the key structural property upon which our algebraic approach derives its results (for a study of this property on a topological setting, see Candeal & Indurain, 1995). Because of the presence of the biasing function in (i′), transitivity of indifference and independence are no longer necessary conditions for these representations. More precisely, this paper shows that, whenever $\succ$ is an interval order (an asymmetric ordering for which $(x \succ y \text{ and } y \succsim z \text{ and } z \succ t) \Rightarrow x \succ t$), there exists a unique function $\gamma(x) : A \to]0,1]$ such that $\sigma(x,y) = \gamma(x)\gamma(y)$. Further, there exists a unique $\alpha \in]0,1]$ such that $\sigma(x,y) = \alpha$ whenever $\succ$ is a semiorder (an interval order for which $x \succ y \succ z \Rightarrow (t \succ z \text{ or } x \succ t)$). Finally, we show that this ratio-scale measurement is extensive, i.e. the scale is additive and verifies property (ii) above (which is stronger than property (ii′)), whenever $\succ$ is a semiorder that verifies the *pseudo-independence* condition

$$\begin{cases} (x \succ y, z \succ t) \Rightarrow x \circ z \succ y \circ t, \\ (x \succsim y, z \succsim t) \Rightarrow x \circ z \succsim y \circ t. \end{cases}$$

In this case, a balance that is not necessarily equally armed provides an enlightening illustration. The statement "$x \succ y$" reflects that the balance tilts towards x independently of the arm on which x is positioned (which means that the balance tilts towards x when it is positioned on the shorter arm) and the statement "$x \circ y$" is interpreted as the positioning of objects x and y in the same pan of the balance. The scale φ measures the mass of the objects and the constant bias α measures the ratio of the length of the shorter arm over the length of the bigger arm. Because the equally armed balance has long been a typical illustration of extensive measurement, the biased balance helps to capture the specificity of biased extensive measurement and of its axioms. For instance, Fig. 1 illustrates the violation of transitive indifference whereas Fig. 2 illustrates the violation of independence.[1]

The biased balance indeed suggested to us that the *homotheticity* condition would adequately replace the traditional independence condition. It is illustrated in Fig. 3.

As stated above, the importance of intransitivity of indifference has long been recognized as a (nearly) inescapable feature of experimental observation and a problematic issue for the mathematical foundations of science (e.g. Poincaré, 1903). Luce (1956) introduced the concept of semiorders to capture the idea that individuals may lack discrimination. In a seminal paper, Scott and Suppes (1958) have proved that finite semiorders can be represented by a function and a constant additive threshold, naturally interpreted as a threshold of indifference (see also Suppes, Krantz, Luce, & Tversky, 1989). A slightly more general notion is the one of interval orders

[1] Note that this is not the only possible interpretation nor the only possible formalization of a biased balance.

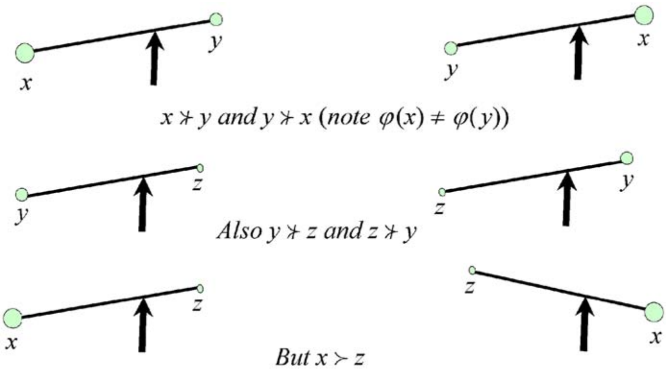

Fig. 1. Violation of transitive indifference with a biased balance.

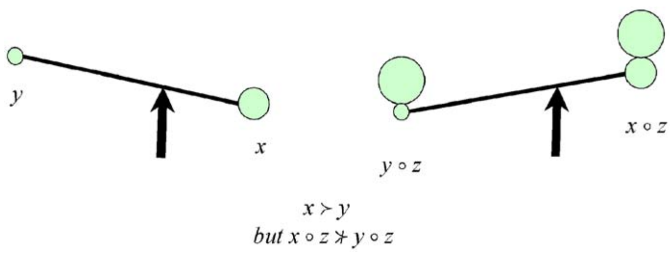

Fig. 2. Violation of independence with a biased balance.

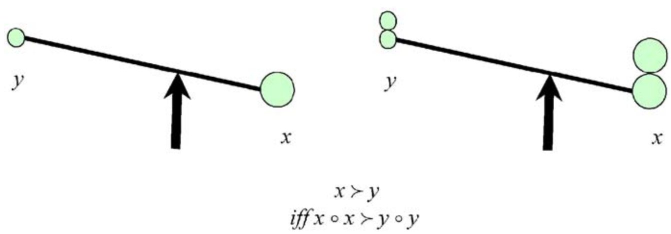

Fig. 3. Homothetic or scale-invariance.

introduced by Fishburn (1973). Several researchers have identified conditions to guarantee the representation of such ordering relations by two functions (e.g. Fishburn, 1973, 1985; Bridges, 1986; Chateauneuf, 1987; Oloriz, Candeal, & Indurain, 1998; Bosi, 2002; Bosi, Candeal, Induráin, & Zudaire, 2005; see also Abbas, 1995). These functions provide for an interval, i.e. a lower and upper value for the measurement of each object, hence their name "interval order".

The representations quoted above do not address the importance of the uniqueness conditions. They do not provide a genuine measurement of objects, either in the sense of a rank ordering (they should prove the existence of a scale unique up to monotonic transformation), or in the sense of an interval scale (they should prove the existence of a scale unique up to positive affine transformation) or in the sense of a ratio-scale like in extensive measurement. In contrast, our theory of biased extensive measurement proves the existence of a ratio-scale that genuinely measure objects even in presence of intransitive indifference, while also providing for a measurement of the distortion of the measurement process. Also, our algebraic approach avoids any of the topological or finiteness assumptions that these representations assume.

Biased extensive measurement has been first introduced in Le Menestrel and Lemaire (2004), where we treat the case of a positive homogeneous substance such as mass in the natural sciences or money in the social sciences. In that restrictive case, we were able to prove both the existence of a ratio-scale and of a unique constant multiplicative bias without transitive indifference and without independence. In Lemaire and Le Menestrel (2006), we provide an abstract algebraic treatment of "homothetic interval orders" and generalize these intermediary results to non-homogeneous structures. Then, the biasing function is not necessarily constant but may vary depending on the objects at hand. In the present paper, we apply our mathematical results to the theory of measurement. We repeat our main results while introducing a supplementary Archimedean axiom that leaves out the consideration of objects that would be assigned a null measurement. From the purely mathematical point of view, this introduces a restriction rather than a novelty but it simplifies the formulations of the theorems and of the proofs. In order to show how this approach allows for the measurement of the distortion of the measurement process, we introduce the notions of indifference sets, of upper and lower indifference thresholds, of tight indifference, of tight upper and lower indifference thresholds, and of progressive refinement of indifference. We illustrate these notions with examples and graphical illustrations.

The rest of this paper is structured as follows. After some preliminaries and a key lemma in Section 2, we first show in Section 3 the existence of a ratio-scale when the ordering relation is a homothetic weak order. In Section 4, we relax the assumption of transitivity of indifference and introduce the biased representation of homothetic interval orders. In Section 5 we recover (in our homothetic context) the more familiar representation of interval orders with two functions, albeit ratio-scales in our case. This leads to a progressive approximation of the measurement of objects. In Section 6, we deal with homothetic semiorders and show that they correspond to the case where the threshold of indifference is a constant proportion of the measurement of objects (Weber's law). In Section 7, we consider the case of homothetic semiorders that have an additive representation, i.e. that satisfy the traditional additive property of extensive measurement (condition (ii) above). We conclude in Section 8.

2. Preliminaries and a key lemma

Let A denote a non-empty set of objects, $x, y, z, t \ldots$ the elements of A, and $\mathbb{N}^*$ the set of positive integers. Our basic algebraic structure consists of the set A together with a map $\mathbb{N}^* \times A \to A, (m, x) \longmapsto mx$ such that $(mm')x = m(m'x)$ and $1x = x$. Such a structure is called a $\mathbb{N}^*$-*set*. In this manner, objects can be replicated with themselves and we interpret mx as the quantity m of object x. Note that the results we obtain for $\mathbb{N}^*$-sets are true (mutatis mutandis) for $\mathbb{R}_+^*$-sets, where $\mathbb{R}_+^*$ denotes the set of positive real numbers. Hence, they are also true for a cone $A = (\mathbb{R}_+^*)^L$ of any dimension L, which is the structure that we use in our examples.

We model the observation of the empirical phenomenon at hand by a binary relation $\succ$ on A. The indifference relation on A is noted $\sim$ and is defined by $x \sim y \Leftrightarrow x \nsucc y$ and $y \nsucc x$. We note $\succsim$ the relation on A defined by $x \succsim y \Leftrightarrow x \succ y$ *or* $x \sim y$.

For all $x, y, z, t \in A$, the relation $\succ$ is said to be *asymmetric* if $x \succ y \Rightarrow y \nsucc x$; *transitive* if $x \succ y \succ z \Rightarrow x \succ z$, *strongly transitive* if ($x \succ y$ *and* $y \succsim z$ *and* $z \succ t$) $\Rightarrow x \succ t$ and *negatively transitive* if $x \nsucc y \nsucc z \Rightarrow x \nsucc z$. Note that the relation $\succ$ is asymmetric if and only if $x \nsucc y \Leftrightarrow y \succsim x$. The relation $\succ$ is called an *interval order* if it is asymmetric and strongly transitive; a *semiorder* if it is an interval order and we have $x \succ y \succ z \Rightarrow$ ($t \succ z$ *or* $x \succ t$); a *weak order* if it is asymmetric and negatively transitive. So we have the implications

weak order $\Rightarrow$ semiorder $\Rightarrow$ interval order.

Note that indifference may fail to be transitive for both an interval order and a semiorder. Also, note that interval orders can been generalized with the notion of biorders (e.g. Doignon, Ducamp, & Falmagne, 1984).

We now introduce the following axioms for a relation $\succ$ on a $\mathbb{N}^*$-set A ($x, y, z, t \in A; m, m', m'' \in \mathbb{N}^*$):

Axiom 1 (*homotheticity*). $\forall(x, y, m)$ we have $x \succ y \Leftrightarrow mx \succ my$;

Axiom 2 (*strong separability*). $\forall(x, y, z)$ such that $x \succ y, \exists(m, m', m'')$ such that $mx \succ m'z, m'z \succsim m''z, m''z \succ my$;

Axiom 3 (*super-Archimedean*).[2] $\forall(x, y)$ such that $x \succ y$, $\exists(m, m')$ such that $m < m'$ and $mx \succ m'y$;

Axiom 4 (*positivity*). $\forall(x, y, m, m')$ such that $m > m'$, we have $x \succ y \Rightarrow mx \succ m'y$;

Axiom 5 (*Archimedean*).[3] $\forall(x, y)$, $\exists m$ such that $mx \succ y$.

We call $\succ$ a *homothetic structure* if it verifies Axioms 1–4. A homothetic structure is called a *homothetic interval order*, a *homothetic semiorder* and a *homothetic weak order* if $\succ$ is, respectively, an interval order, a semiorder and a weak order. In this paper, we will further suppose that a homothetic structure is Archimedean, i.e. that Axiom 5 always holds, excluding the objects that would be assigned a null utility.[4] This is the axiom which is omitted in Lemaire and Le Menestrel (2006).

We now introduce the basic tools of our algebraic approach.

Let A be a homothetic structure and define the (non-empty) subsets of $\mathbb{Q}_+^*$, where $\mathbb{Q}_+^*$ denotes the set of positive rational numbers:

$$\mathcal{Q}_{x,y} = \{mn^{-1} : m, n \in \mathbb{N}^*, mx \succsim ny\},$$

$$\mathcal{P}_{x,y} = \{mn^{-1} : m, n \in \mathbb{N}^*, mx \succ ny\}.$$

Let $r_{x,y} = \inf_{\mathbb{R}_{\geqslant 0}} \mathcal{Q}_{x,y}$ and $s_{x,y} = \inf_{\mathbb{R}_{\geqslant 0}} \mathcal{P}_{x,y}$. For non-empty subsets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^*$, let $\mathcal{U}^{-1} = \{u^{-1}, u \in \mathcal{U}\}$ and $\mathcal{U}\mathcal{V} = \{uv, u \in \mathcal{U}, v \in \mathcal{V}\}$. If $\succ$ is asymmetric, the $\mathbb{Q}_+^*$ is partitioned as $\mathbb{Q}_+^* = \mathcal{Q}_{x,y} \cup \mathcal{P}_{y,x}^{-1} = \mathcal{Q}_{y,x}^{-1} \cup \mathcal{P}_{x,y}$ with $\mathcal{Q}_{x,y} \cap \mathcal{P}_{y,x}^{-1} = \mathcal{Q}_{y,x}^{-1} \cap \mathcal{P}_{x,y} = \varnothing$.

We now prove the following useful lemma.

Lemma 1. *If $\succ$ is an interval order on a homothetic structure, then for all $x, y, a \in A$ we have $\mathcal{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$ with $s_{x,y} > 0$, $\mathcal{Q}_{y,x} = \mathbb{Q}_{\geqslant r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$, and $\mathcal{P}_{x,y} = \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$.*

Proof. Let $x, y \in A$. If $q \in \mathcal{P}_{x,y}$, we have (Axiom 4) $\mathbb{Q}_{\geqslant q} \subset \mathcal{P}_{x,y}$. If $q \in \mathbb{Q}_{>s_{x,y}}$, then by definition of $s_{x,y}$, there exists $q' \in \mathcal{P}_{x,y}$ such that $s_{x,y} \leqslant q' < q$. Hence, we have $\mathbb{Q}_{>s_{x,y}} \subset \mathcal{P}_{x,y}$. From Axiom 2, we have $s_{x,y} \in \mathbb{Q} \Rightarrow s_{x,y} \notin \mathcal{P}_{x,y}$. Thus, $\mathcal{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$. Since $\mathcal{Q}_{y,x}^{-1} = \mathbb{Q}_+^* \backslash \mathcal{P}_{x,y}$ is non-empty (Axiom 5), we have $s_{x,y} > 0$. Since $\mathcal{Q}_{y,x}^{-1} = \mathbb{Q}_+^* \backslash \mathcal{P}_{x,y}^{-1} =]0, s_{x,y}]$, we have $\mathcal{Q}_{y,x} = \mathbb{Q}_{\geqslant s_{x,y}^{-1}}$ and $r_{y,x} = s_{x,y}^{-1}$. From strong transitivity of $\succ$ and Axiom 1, we have the inclusion $\mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y} \subset \mathcal{P}_{x,y}$; and from Axioms 2 and 1, we have the inclusion $\mathcal{P}_{x,y} \subset \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$. $\square$

3. Homothetic weak orders

Suppose that the binary relation $\succ$ is a weak order, i.e. that there is no lack of discrimination in the measurement process. In our homothetic setting, we show that we can prove the existence of a ratio-scale. Our situation is here a little bit different than the traditional extensive measurement on extensive structure in the sense that no concatenation operation beyond simple replication is assumed among objects. As a result, we obtain a ratio scale φ that does not necessarily verify additivity, as it will be shown in the following example. Note also that a similar result appeared as Theorem 9 of Krantz et al. (1971, p. 104). The formulation above has the interest of building directly on the homotheticity condition, allowing more simplicity in the proof and providing the necessary generality for the subsequent results without transitive indifference.

[2] Note that, in Lemaire and Le Menestrel (2006) and Le Menestrel and Lemaire (2004, 2006), we use the terminology "Archimedean" for Axiom 3. The terminology used in the present paper has been suggested by the editor and seems more appropriate (see in particular Candeal, De Miguel, & Indurain, 1997, De Miguel, Candeal, & Indurain, 1996).

[3] For the terminology, Cf. note 2 above.

[4] Remark that A can be (uniquely) decomposed in a disjoint union of homogeneous subsets (a set S is called *homogeneous* if for all $x, y \in S$, there exists $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx = ny$, see Le Menestrel & Lemaire, 2004) called homogeneous classes in A. When Axiom 5 holds, each homogeneous class C in A is an infinite denumerable set (in particular, C cannot be a finite cycle group). Moreover, Axiom 5 forces the map φ of Theorems 1–6 below to verify the following condition: for each homogeneous class C in A, the restriction $\varphi_{|C}$ is an injective map.

Theorem 1. *Let A be a non-empty $\mathbb{N}^*$-set endowed with an Archimedean binary relation $\succ$. The two following conditions are equivalent* $(x, y \in A; m \in \mathbb{N}^*)$:

(i) *There exists a function* $\varphi : A \to \mathbb{R}_+^*$ *such that* $\forall (x, y, m)$ *we have*

$$\begin{cases} x \succ y \Leftrightarrow \varphi(x) > \varphi(y), \\ \varphi(mx) = m\varphi(x). \end{cases}$$

(ii) *The relation $\succ$ is a homothetic weak order.*

Moreover, if $\succ$ is a homothetic weak order, the function φ of (i) *is unique up to multiplication by a positive number, i.e. it is a ratio scale.*

Proof. Implication (i) $\Rightarrow$ (ii) is clear. Suppose $\succ$ is a homothetic weak order. From Lemma 1, we have $x \succ y \Leftrightarrow s_{x,y} < 1$. Let us prove that $s_{x,x} = 1 = r_{x,x}$. Suppose that $s_{x,x} \neq 1$. Then, $r_{x,x} < 1$. Therefore, there exist $m < n \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx \gtrsim nx$. By Axioms 1 and 4, we thus have $m^2 x \gtrsim mnx \gtrsim n^2 x$. By transitivity of $\gtrsim$, we have $m^2 x \gtrsim n^2 x$. Hence, $m^k x \gtrsim n^k x$ for all $k \in \mathbb{N}^*$. Because $\lim_{k \to +\infty} (\frac{m}{n})^k = 0$, we obtain $r_{x,x} = 0$. Hence, $\mathscr{P}_{x,x} = \varnothing$ which contradicts Axiom 5. Choose an element $a \in A$ and let $\varphi : A \to \mathbb{R}_+^*$ be the function defined by $\varphi(x) = r_{a,x}$. Clearly, we have $\varphi(mx) = m\varphi(x)$. Let us prove that $x \succ y \Leftrightarrow \varphi(x) > \varphi(y)$. From the equality $\mathscr{P}_{x,x} = \mathscr{P}_{x,a}\ \mathscr{Q}_{a,a}\mathscr{P}_{a,x}$, we obtain $s_{x,x} = s_{x,a} r_{a,a} s_{a,x}$, that is $s_{a,x} = s_{x,a}^{-1} = r_{a,x}$. Hence $s_{x,y} = s_{x,a} r_{a,a} s_{a,y} = r_{a,x}^{-1} r_{a,y}$. Since $x \succ y \Leftrightarrow s_{x,y} < 1$, we have $x \succ y \Leftrightarrow \varphi(x) > \varphi(y)$.

Now let $\psi : A \to \mathbb{R}_+^*$ be another function verifying (i). Let $\lambda : A \to \mathbb{R}_+^*$ be the function defined by $\lambda(x) = \varphi(x)^{-1}\psi(x)$. Suppose there exist two elements $x, y \in A$ such that $\lambda(x) \neq \lambda(y)$. By symmetry, we can assume $\lambda(x) > \lambda(y)$. Let $\alpha = \lambda(y)\lambda(x)^{-1} < 1$. Then by density, there exists a $q \in \mathbb{Q}_+^*$ such that $\alpha\varphi(y)\varphi(x)^{-1} < q < \varphi(y)\varphi(x)^{-1}$. In other words, we have $\psi(y) < q\psi(x)$ and $q\varphi(x) < \varphi(y)$, which is impossible. Hence, λ is a constant map. □

Let $\succ$ be an Archimedean homothetic weak order on a $\mathbb{N}^*$-set A. We chose a function verifying condition (i) of Theorem 1 and we say that φ *represents* $\succ$.

For $x \in A$, we note φ_x the *isocontour containing* x, defined by $\varphi_x = \{y \in A : \varphi(y) = \varphi(x)\}$. Note that φ_x does not depend on φ.

We now illustrate this ratio-scale measurement with three equally spaced isocontours. In this illustration, like in all other examples in this paper, our basic structure is a two-dimensional space. It can be interpreted as stimuli composed of two attributes or bundles of goods that consists in objects made of a quantity x_1 of good X_1 and x_2 of good X_2. In order to ease the graphical and numerical illustrations, which may feature non integer values, we consider real-valued quantities of goods. Formally, A is the set $\{(x_1X_1, x_2X_2) : x_1, x_2 \in \mathbb{R}_+^*\}$ endowed with the structure of $\mathbb{R}_+^*$-set given by the map $\mathbb{R}_+^* \times A \to A$, $(\lambda, (x_1X_1, x_2X_2)) \mapsto (\lambda x_1 X_1, \lambda x_2 X_2)$.

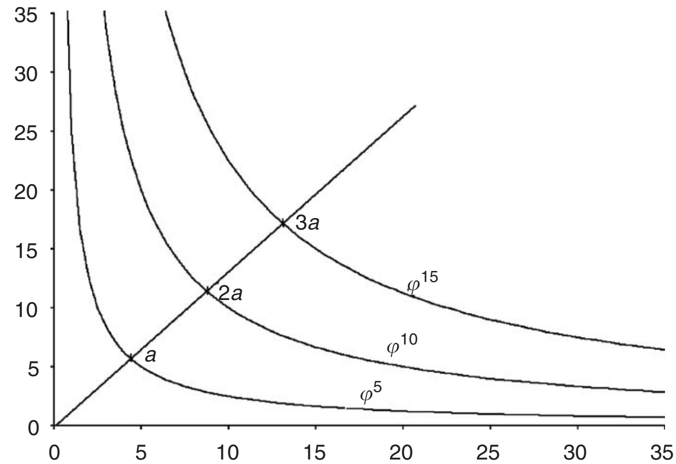

Fig. 4. Three equally-spaced isocontours of a homothetic weak order.

Example 1. Consider the function $x = (x_1X_1, x_2X_2) \mapsto \varphi(x) = x_1^{1/2} x_2^{1/2}$ and define $\succ$ by $x \succ y \Leftrightarrow \varphi(x) > \varphi(y)$ for all $x, y \in A$. With the quantity x_1 in abscissae and the quantity x_2 as ordinate, Fig. 4 plots three equally spaced isocontours $\varphi^5 = \{x : \varphi(x) = 5\}, \varphi^{10} = \{x : \varphi(x) = 10\}, \varphi^{15} = \{x : \varphi(x) = 15\}$. In this manner, there is as much difference between the values of objects belonging to φ^5 and φ^{10} than between the values of objects belonging to φ^{10} and φ^{15} (preservation of differences). Moreover, the value of objects belonging to φ^{10} is twice the value of objects belonging to φ^5 (valuation of ratios). We illustrate this with the objects $a \in \varphi^5, 2a \in \varphi^{10}$, and $3a \in \varphi^{15}$.

4. Homothetic interval orders, biased representation

Suppose now the binary relation $\succ$ is an interval order. Then, indifference is no more transitive, for instance because of a lack of discrimination in the measurement process. We show that we can still prove the existence of a ratio-scale for the measurement of objects. Moreover, we can characterize the bias, or distortion factor, that induces the binary relation to depart from the underlying weak order. This bias appears as a function of each object, that is positive and smaller or equal to 1, and that is uniquely characterized by the underlying homothetic interval order.

Because of the intransitivity of indifference, isocontours show a sort of "thickness" and objects may be indifferent without having the same measure. We give a precise characterization of the indifference set containing a given object, making precise the idea that two objects having the same measure may have different indifference sets because they differ qualitatively by their biasing function. Also, we propose a measure of the threshold of indifference for a given object by introducing and illustrating the notions of upper and lower indifference thresholds. We illustrate these notions with a two-dimensional space.

Theorem 2. *Let A be a non-empty $\mathbb{N}^*$-set endowed with an Archimedean binary relation $\succ$. The two following conditions* *are equivalent* ($x, y \in A; m \in \mathbb{N}^*$):

(i) *there exist two functions* $\varphi : A \to \mathbb{R}_+^*$ *and* $\gamma : A \to]0,1]$ *such that* $\forall(x, y, m)$ *we have*

$$\begin{cases} x \succ y \Leftrightarrow \gamma(x)\gamma(y)\varphi(x) > \varphi(y), \\ \varphi(mx) = m\varphi(x), \\ \gamma(mx) = \gamma(x). \end{cases}$$

(ii) *The relation* $\succ$ *is a homothetic interval order*.

Moreover, if $\succ$ *is a homothetic interval order, the pair* (φ, γ) *of* (i) *is unique up to multiplication of* φ *by a positive number*.

Proof. The implication (i) $\Rightarrow$ (ii) is easy to verify. Suppose $\succ$ is a homothetic interval order. Using Lemma 1, we have $x \succ y \Leftrightarrow s_{x,y} < 1$ (and also $x \succ y \Leftrightarrow r_{y,x} > 1$). Let $\succ_0$ be the binary relation on A defined by $x \succ_0 y \Leftrightarrow s_{y,x} > s_{x,y}$; i.e. by $x \succ_0 y \Leftrightarrow \mathscr{P}_{x,y} \supsetneq \mathscr{P}_{y,x}$. Since $s_{y,mx} = ms_{y,x}$ and $s_{mx,y} = m^{-1}s_{x,y}$, $\succ_0$ is Archimedean. We now prove that $\succ_0$ is a homothetic weak order. Let $\sim_0$ be the indifference relation associated with $\succ_0$. Thus we have $x \sim_0 y \Leftrightarrow s_{x,y} = s_{y,x} \Leftrightarrow \mathscr{P}_{x,y} = \mathscr{P}_{y,x}$. By usingthe equalities $s_{x,z} = s_{x,y}r_{y,y}s_{y,z}$ and $s_{z,x} = s_{z,y}r_{y,y}s_{y,x}$, we obtain the transitivity of $\sim_0$: if $x \sim_0 y$ and $y \sim_0 z$, then $x \sim_0 z$. Hence $\succ_0$ is negatively transitive; in particular, it is a weak order. For all $x, y \in A$ and all $m, n \in \mathbb{N}^*$, we have $\mathscr{P}_{mx,ny} = \frac{n}{m}\mathscr{P}_{x,y}$, and therefore $s_{mx,ny} = \frac{n}{m}s_{x,y}$. We then easily deduce that $\succ_0$ satisfies Axioms 1, 3 and 4. It remains to verify that $\succ_0$ satisfies Axiom 2. Let $x, y \in A$ such that $x \succ_0 y$. Since $s_{x,y} = s_{x,z}r_{z,z}s_{z,y}$ and $s_{y,x} = s_{y,z}r_{z,z}s_{z,x}$, we have $s_{x,z}s_{z,y} < s_{y,z}s_{z,x}$. Hence there exist $p, m, n \in \mathbb{N}^*$ with $m > n$ such that $(\frac{m}{p})^2 \frac{s_{x,z}}{s_{z,x}} < 1 < (\frac{n}{p})^2 \frac{s_{y,z}}{s_{z,y}}$and $(pn^{-1})^2 s_{z,y} < s_{y,z}$. Then we have $s_{px,mz} < s_{mz,px}$ and $s_{nz,py} < s_{py,nz}$, i.e. $px \succ_0 mz \gtrsim_0 nz \succ_0 py$. We have thus proven that $\succ_0$ is a homothetic weak order.

By Theorem 1, we can choose a function $\varphi : A \to \mathbb{R}_+^*$ such that $\varphi(mx) = m\varphi(x)$ and $x \succ_0 y \Leftrightarrow \varphi(x) > \varphi(y)$. Let $\sigma(x, y) : A \times A \to \mathbb{R}_+^*$ be the function defined by $\sigma(x, y) = r_{y,x}\varphi(x)^{-1}\varphi(y)$. Since $x \succ y \Leftrightarrow r_{y,x} > 1$, by construction we have $x \succ y \Leftrightarrow \sigma(x, y)\varphi(x) > \varphi(y)$. And returning to the definition of $\succ_0$, we obtain

$$\varphi(x) > \varphi(y) \Leftrightarrow \sigma(x, y)^{1/2}\varphi(x) > \sigma(y, x)^{1/2}\varphi(y).$$

This implies $\sigma(x, y) = \sigma(y, x)$. For $x, y, x', y' \in A$, we have $r_{y,x} = r_{y',x}s_{y',y'}r_{y,y'}$, hence $r_{y,x}r_{y',x'} = r_{y',x}(r_{y',x'}s_{y',y'}r_{y,y'}) = r_{y',x}r_{y,x'}$. Hence, we have $\sigma(x, y)\sigma(x', y') = \sigma(x, y')\sigma(x', y)$. Therefore, we have $\sigma(x, y) = \gamma(x)\gamma(y)$ with $\gamma(x) = \sigma(x, x)^{1/2}$. Since $\sigma(mx, m'y) = \sigma(x, y)$, we have $\gamma(mx) = \gamma(x)$. The uniqueness of φ up to multiplication by a positive number (Theorem 1) implies the uniqueness of γ. □

Let $\succ$ be an Archimedean homothetic interval order on a $\mathbb{N}^*$-set A. We chose a pair (φ, γ) verifying condition (i) of Theorem 2 and we say that (φ, γ) *represents* $\succ$. When $\gamma = 1$ (i.e. the constant function $x \mapsto 1$), we recover Theorem 1: indifference is transitive and $\succ$ is a weak order.

For $x \in A$, we note I_x the *indifference set containing* x, defined by $I_x = \{y \in A : y \sim x\}$. Note that I_x does not depend on φ. And because of the symmetry of $\sim$ (i.e. $x \sim y \Leftrightarrow y \sim x$), we have $y \in I_x \Leftrightarrow x \in I_y$. Moreover, if $\succ$ is a weak order (i.e. if $\gamma = 1$), then I_x coincides with the isocontour $\varphi_x = \{y \in A : \varphi(y) = \varphi(x)\}$. When indifference is not transitive, indifference sets show a threshold of indifference. We have

$$y \in I_x \Leftrightarrow \gamma(x)\gamma(y)\varphi(x) \leqslant \varphi(y) \leqslant \gamma(x)^{-1}\gamma(y)^{-1}\varphi(x).$$

This can also be written

$$x \sim y \Leftrightarrow \varphi(x) = \varphi(y) \overset{\times}{\div} \gamma(x)\gamma(y).$$

Therefore, two objects with the same measure may have different indifference sets. For $x \in A$, we note $\mathfrak{I}_x$ the subset of $\mathbb{Q}_+^*$ defined by $\mathfrak{I}_x = \{\frac{m}{n} : mx \sim nx\}$. Because of the symmetry of $\sim$, we have $\mathfrak{I}_x^{-1} = \mathfrak{I}_x$. Let $\bar{\mathfrak{I}}_x$ denote the closure of $\mathfrak{I}_x$ in $\mathbb{R}$ for the usual topology. We deduce from Theorem 2 that $\bar{\mathfrak{I}}_x$ coincides with the closed interval $[\gamma(x)^2 : \gamma(x)^{-2}]$. We let $\delta_x^+ = \gamma^{-2}(x)$ and $\delta_x^- = (\delta_x^+)^{-1}$. Thus, we have $\bar{\mathfrak{I}}_x = [\delta_x^-, \delta_x^+]$. We propose to call δ_x^+ the *upper indifference threshold at* x, and δ_x^- the *lower indifference threshold at* x. We illustrate these concepts in the following example.

Example 2. Consider the ratio-scale $x = (x_1X_1, x_2X_2) \mapsto \varphi(x) = x_1^{1/2}x_2^{1/2}$ and consider a factor $\gamma(x) = \frac{\lambda x_1 + \mu x_2}{x_1 + x_2}$ with $\lambda, \mu \leqslant 1$ that biases bundle x depending on the relative quantities of objects X_1 and X_2. The binary relation $\succ$ defined by $x \succ y \Leftrightarrow \gamma(x)\varphi(x) > \gamma(y)^{-1}\varphi(y)$ for all $x, y \in A$ is a homothetic interval order. Letting $\lambda = 0.95$ and $\mu = 0.80$, Fig. 5 shows object $a = (25X_1, 9X_2)$ and $b = (9X_1, 25X_2)$ with $\varphi(a) = \varphi(b) = 15$. Their identical isocontour φ^{15} appears in bold line and their distinct indifference sets I_a and I_b are delimited by the plain and dotted lines, respectively. Note that since $\gamma(a) > \gamma(b)$, we have $I_a \subset I_b$. We also show the lower and upper indifference thresholds of objects a and b. Since A is a $\mathbb{R}_+^*$-set; for $x \in A$, δ_x^+ coincides with the $\sup\{\lambda \in \mathbb{R}_+^* : \lambda x \sim x\}$, and δ_x^- coincides with the $\inf\{\lambda \in \mathbb{R}_+^* : \lambda x \sim x\}$. Numerically, we have $\delta_a^- = 0.83$ which corresponds to a decrease of 17% and $\delta_a^+ = 1.42$ which corresponds to an increase of 21%. Also, $\delta_b^- = 0.71$ (−29%) and $\delta_b^+ = 1.42$ (+42%).

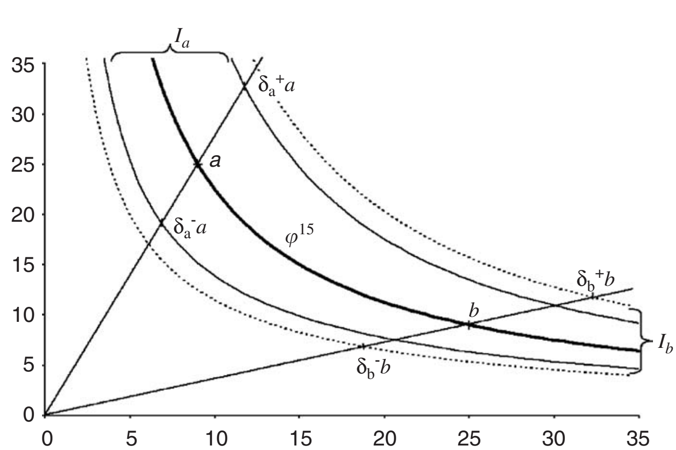

Fig. 5. Two indifference sets of a homothetic interval order.

5. Homothetic interval orders, tight representation with two functions

In this section, we rejoin the classical representation of an interval order $\succ$ that consists in finding two real-valued functions φ_1 and φ_2, with $\varphi_1 \leqslant \varphi_2$, such that $x \succ y \Leftrightarrow \varphi_1(x) > \varphi_2(y)$. For a given homothetic interval order $\succ$, we construct such a canonical two-function representation (φ_1, φ_2) directly from two weak orders associated with $\succ$ and where both φ_1 and φ_2 are ratio-scales. This ensures that the pair (φ_1, φ_2) is unique up to multiplication by a positive number.

Let A be a non-empty set $\mathbb{N}^*$-set endowed with an Archimedean homothetic interval order $\succ$. We define the following three binary relations:

- $x \succ_0 y \Leftrightarrow \varphi(x) > \varphi(y)$ for one (i.e. for any) pair (φ, γ) verifying condition (i) of Theorem 2,
- $x \succ_1 y \Leftrightarrow (mx \succ z$ and $z \succsim my$, for some $(z, m) \in A \times \mathbb{N}^*)$,
- $x \succ_2 y \Leftrightarrow (mx \succsim z$ and $z \succ my$, for some $(z, m) \in A \times \mathbb{N}^*)$.

Note that $x \succ y \Rightarrow (x \succ_1 y$ *and* $x \succ_2 y)$. Hence, since $\succ$ is strongly non-empty, so is $\succ_i (i = 1, 2)$. The relation $\succ_0$ is clearly a homothetic weak order. The following corollary shows the same is true for both $\succ_1$ and $\succ_2$.

Corollary 1. *Let A be a non-empty $\mathbb{N}^*$-set endowed with an Archimedean homothetic interval order $\succ$. Then for $i = 1, 2, \succ_i$ is a homothetic weak order.*

Proof. Since $\succ$ is an Archimedean homothetic interval order, let (φ, γ) be a pair representing $\succ$ (Theorem 2). Choose an element $a \in A$. Let $x, y \in A$ such that $x \succ_1 y$, and let $(z, m) \in A \times \mathbb{N}^*$ such that $mx \succ z$ and $z \succsim my$. We have $\gamma(x)\gamma(z)\varphi(mx) > \varphi(z)$ and $\varphi(z) \geqslant \gamma(x)\gamma(y)\varphi(my)$. Therefore, we have

$$r_{z,x}\frac{\varphi(z)}{\varphi(x)}\varphi(mx) > \varphi(z) \geqslant r_{z,y}\frac{\varphi(z)}{\varphi(y)}\varphi(my),$$

hence $r_{z,x} > r_{z,y}$. Moreover, we have $r_{z,x} = r_{z,a}s_{a,a}r_{a,x}$ and $r_{z,y} = r_{z,a}s_{a,a}r_{a,y}$. Therefore, we have $r_{a,x} > r_{a,y}$.

Conversely, let $x, y \in A$ such that $r_{a,x} > r_{a,y}$. Then, there exist $m, n \in \mathbb{N}^*$ such that $r_{a,x} > \frac{n}{m} \geqslant r_{a,y}$. Since $\frac{1}{n} r_{a,x} = r_{na,x}$, we have $mr_{na,x} > 1 \geqslant mr_{na,y}$, i.e. $\gamma(x)\gamma(a)\varphi(mx) > \varphi(na)$ and $\varphi(na) \geqslant \gamma(y)\gamma(a)\varphi(my)$. Thus, we have $mx \succ na$ and $na \succsim my$, i.e. $x \succ_1 y$. Therefore, we have proven that the function $\varphi_1 : A \rightarrow \mathbb{R}_+^*$ such that $\varphi_1(x) = r_{a,x}$ represents $\succ_1$. Since we clearly have $\varphi_1(mx) = m\varphi_1(x)$, $\succ_1$ is a homothetic weak order.

Let $x, y \in A$ such that $x \succ_2 y$, and let $(z, m) \in A \times \mathbb{N}^*$ such that $mx \succ z$ and $z \succsim my$. We have $\varphi(mx) \geqslant \gamma(x)\gamma(z)\varphi(z)$ and $\gamma(x)\gamma(y)\varphi(z) > \varphi(my)$. Therefore, we have $\gamma(x)^{-1}\gamma(z)^{-1}\varphi(mx) \geqslant \varphi(z) > \gamma(x)^{-1}\gamma(z)^{-1}\varphi(my)$. Hence, we obtain $s_{a,x} > s_{a,y}$. As for $\succ_1$, we show conversely that, for all $x, y \in A$ such that $s_{a,x} > s_{a,y}$, we have $x \succ_2 y$. Therefore, the function $\varphi_2 : A \rightarrow \mathbb{R}_+^*$ such that $\varphi_2(x) = s_{a,x}$ represents $\succ_2$ and $\succ_2$ is a homothetic weak order. $\square$

We can now prove the following theorem:

Theorem 3. *Let A be a non-empty $\mathbb{N}^*$-set endowed with an Archimedean binary relation $\succ$. The two following conditions areе equivalent $(x, y \in A; m \in \mathbb{N}^*)$:*

(i) *There exist two functions $\varphi_1, \varphi_2 : A \rightarrow \mathbb{R}_+^*$ such that $\varphi_1 \leqslant \varphi_2$ and $\forall (x, y, m)$ we have*

$$\begin{cases} x \succ y \Leftrightarrow \varphi_1(x) > \varphi_2(y), \\ \varphi_1(mx) = m\varphi_1(x), \\ \varphi_2(mx) = m\varphi_2(x). \end{cases}$$

(ii) *The relation $\succ$ is a homothetic interval order.*

Moreover, if $\succ$ is a homothetic interval order, the pair (φ_1, φ_2) of (i) *is unique up to multiplication of φ_1 and φ_2 by a positive number* (*i.e., up to replacing it by $(\lambda\varphi_1, \lambda\varphi_2)$ for a $\lambda > 0$*)*; for $i = 1, 2$, the function φ_i represents $\succ_i$; and the pair $(\varphi, \gamma) = ((\varphi_1\varphi_2)^{1/2}, (\frac{\varphi_1}{\varphi_2})^{1/2})$ verifies the condition* (i) *of Theorem 2.*

Proof. The implication (i) $\Rightarrow$ (ii) is easy to verify. Suppose $\succ$ is a homothetic interval order. Choose an element $a \in A$ and let $\varphi_1, \varphi_2 : A \rightarrow \mathbb{R}_+^*$ be the functions defined by $\varphi_1(x) = s_{a,a}r_{a,x}$ and $\varphi_2(x) = s_{a,x}$. For $i = 1, 2$, we clearly have $\varphi_i(mx) = m\varphi_i(x)$. For all $x, y \in A$, we have

$$\begin{aligned} x \succ y &\Leftrightarrow r_{y,x} > 1 \\ &\Leftrightarrow r_{y,a}s_{a,a}r_{a,x} > 1 \\ &\Leftrightarrow s_{a,a}r_{a,x} > s_{a,y} \\ &\Leftrightarrow \varphi_1(x) > \varphi_2(y). \end{aligned}$$

From Corollary 1, we already know that, for $i = 1, 2$, φ_i *represents* $\succ_i$. Moreover, for all $x, y \in A$, we have

$$\begin{aligned} s_{y,x} > s_{x,y} &\Leftrightarrow s_{y,a}r_{a,a}s_{a,x} > s_{x,a}r_{a,a}s_{a,y} \\ &\Leftrightarrow r_{a,x}s_{a,x} > r_{a,y}s_{a,y} \\ &\Leftrightarrow (\varphi_1\varphi_2)(x) > (\varphi_1\varphi_2)(y). \end{aligned}$$

Hence, $(\varphi_1\varphi_2)$ represents $\succ_0$. Therefore, $\varphi = (\varphi_1\varphi_2)^{1/2}$ represents $\succ_0$. Clearly, we have $\varphi(mx) = m\varphi(x)$. And from the proof of Theorem 2, we have $\gamma(x)^2 = r_{x,x} = (\frac{\varphi_1}{\varphi_2})(x)$.

To show the uniqueness property, take (φ_1, φ_2) and (φ_1', φ_2') two pairs of functions verifying condition (i) of Theorem 3. For $i = 1, 2$, let $\lambda_i : A \rightarrow \mathbb{R}_+^*$ be the function defined by $\lambda_i(x) = \frac{\varphi_i'(x)}{\varphi_i(x)}$. Take $x, y \in A$ and let $\alpha = \frac{\lambda_1(x)}{\lambda_2(x)} \in \mathbb{R}_+^*$. If $\alpha > 1$, then there exists $q \in \mathbb{Q}_+^*$ such that $q \leqslant \frac{\varphi_2(y)}{\varphi_1(x)} < \alpha q$. Let us write $q = \frac{m}{n}$ with $m, n \in \mathbb{N}^*$. We then have $\varphi_1(mx) \leqslant \varphi_2(ny)$ and $\varphi_1'(mx) > \varphi_2'(ny)$, contradiction. If $\alpha < 1$, then there exists $q' \in \mathbb{Q}_+^*$such that $\alpha q' < \frac{\varphi_2(y)}{\varphi_1(x)} \leqslant q'$. Write $q' = \frac{m'}{n'}$ with $m', n' \in \mathbb{N}^*$. We then have $\varphi_1'(m'x) \leqslant \varphi_2'(n'y)$ and $\varphi_1(m'x) > \varphi_2(n'y)$, contradiction. Hence, $\alpha = 1$. Since this is true for any $x, y \in A$, there exists a constant $\lambda \in \mathbb{R}_+^*$ such that $\lambda = \lambda_1 = \lambda_2$. $\square$

Let $\succ$ be an Archimedean homothetic interval order on a $\mathbb{N}^*$-set A. We choose two functions (φ_1, φ_2) verifying condition (i) of Theorem 3 and we say that (φ_1, φ_2) *represents* $\succ$. We put $(\varphi, \gamma) = ((\varphi_1\varphi_2)^{1/2}, (\frac{\varphi_1}{\varphi_2})^{1/2})$.

For $x \in A$, we note J_x the set $\{y : \varphi_1(x) \leqslant \varphi(y) \leqslant \varphi_2(x)\}$, and we propose to call it the *tight indifference set containing* x. Because of the uniqueness property of Theorem 2, J_x does not depend on the choice of the pair (φ_1, φ_2).

For $x \in A$, we have the inclusion (in general strict) $J_x \subset I_x$. By construction, if $y, z \in J_x$, then $y \sim z$; a property that is not verified by indifference sets. Note also that we may have $y \in J_x$ but $x \notin J_y$. More precisely, for $x, y \in A$, we have $x \sim y$ if and only if $J_x \cap J_y \neq \emptyset$, i.e. if and only if the intersection of the two closed intervals $\varphi(J_x)$ and $\varphi(J_y)$ is non-empty.

Let $\approx$ be the binary relation on A defined as follows: $x \approx y$ if and only if $x \in J_y$ and $y \in J_x$. It is clearly symmetric, and we call it the *tight indifference* relation associated with $\succ$. For $x \in A$, we note $\mathfrak{J}_x$ the subset of $\mathbb{Q}_+^*$ defined by $\mathfrak{J}_x = \{\frac{m}{n} : mx \approx nx\}$. We have $\mathfrak{J}_x = \mathfrak{J}_x^{-1}$. Let $\bar{\mathfrak{J}}_x$ denote the closure of $\mathfrak{J}_x$ in $\mathbb{R}$ for the usual topology. We deduce from Theorem 2 (or Theorem 3) that $\bar{\mathfrak{J}}_x$ coincide with the closed interval $[\gamma(x), \gamma(x)^{-1}]$. We put $\tau_x^+ = \gamma(x)^{-1}$ and $\tau_x^- = (\tau_x^+)^{-1}$. Thus, we have $\bar{\mathfrak{J}}_x = [\tau_x^-, \tau_x^+]$. We call τ_x^+ the *upper tight indifference threshold at* x, *and* τ_x^- the *lower tight indifference threshold at* x. We illustrate these concepts in the following example.

Example 3. Consider the two functions $x = (x_1X_1, x_2X_2) \mapsto \varphi_1(x) = \lambda x_1 + x_2$ and $x = (x_1X_1, x_2X_2) \mapsto \varphi_2(x) = \mu x_1 + x_2$ where $0 < \lambda \leqslant \mu$ and define $\succ$ by $x \succ y \Leftrightarrow \varphi_1(x) > \varphi_2(y)$. The relation $\succ$ is a homothetic interval order and we recover the formulation $x \succ y \Leftrightarrow \gamma(x)\varphi(x) > \gamma^{-1}(y)\varphi(y)$ of Theorem 2 with $\gamma(x) = (\lambda x_1 + x_2)^{1/2}(\mu x_1 + x_2)^{-1/2}$ and $\varphi(x_1X_1, x_2X_2) = (\lambda x_1 + x_2)^{1/2}(\mu x_1 + x_2)^{1/2}$. Letting $\lambda = \frac{5}{14}$ and $\mu = \frac{6}{7}$, Fig. 6 shows object $a = (14, 4)$ of measure $\varphi(a) = 12$ and its isocontour $\varphi^{12} = \{x : \varphi(x) = 12\}$ in bold line. In plain lines appears the tight indifference set J_a and in dotted lines appears the indifference set I_a. We also depict thelower and upper indifference and tight indifference thresholds of object a. Since A is a $\mathbb{R}_+^*$-set; for $x \in A$, τ_x^+ coincides with the $\sup\{\lambda \in \mathbb{R}_+^* : \lambda x \approx x\}$, and τ_x^- coincides with the $\inf\{\lambda \in \mathbb{R}_+^* : \lambda x \approx x\}$. Numerically, we have $\delta_a^- = \frac{9}{16}$ and $\delta_a^+ = \frac{16}{9}$. Also, $\tau_a^- = \frac{3}{4}$ and $\tau_a^+ = \frac{4}{3}$.

The notions of indifference sets and tight indifference set that we introduced suggest that a series of progressively tighter and tighter indifference relations can be constructed until we eventually reach the equivalence relation among objects with identical measure. It is indeed the case and we now formalize this intuition.

For a homothetic interval order $\succ$ represented by a pair (φ, γ) (Theorem 2) and for $k \in \mathbb{N}^*$, we define the homothetic interval order $\succ^k$ by the pair $(\varphi, \gamma^{1/k})$ and note $\sim^k$ its associated indifference relation. We have $\succ^1 = \succ$ and $\succ^k \subset \succ^{k+1}$ (i.e. $\succ^{k+1}$ is thinner than $\succ^k$): $x \succ^k y \Rightarrow x \succ^{k+1} y$. If $x \sim^k y$, we say that x and y are *k-indifferent*.

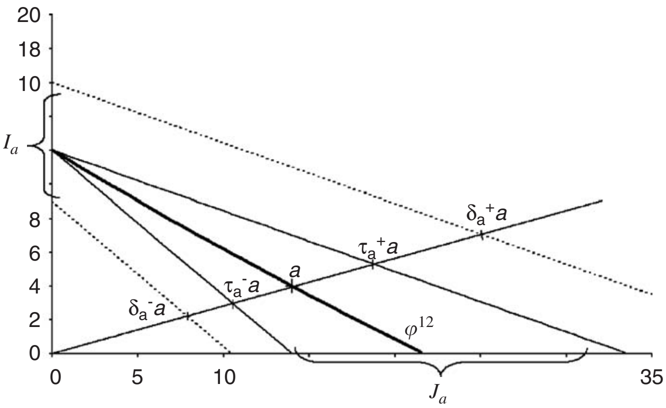

Fig. 6. A tight indifference set and an indifference set of a homothetic interval order.

Theorem 4. *Let A be a non-empty* $\mathbb{N}^*$*-set endowed with an Archimedean homothetic interval order* $\succ$. *Then for* $x, y \in A$ *such that* $x \sim y$, *either there exists* $k \in \mathbb{N}^*$ *such that* $x \sim^k y$ *and* $x \nsim^{k+1} y$, *or x and y have the same measure* (*i.e.* $y \in \varphi_x$).

Proof. If $\succ$ is a weak order, then there is nothing to prove: $x \sim y \Leftrightarrow y \in \varphi_x$, and $\succ^k = \succ$ for all $k \in \mathbb{N}^*$.So we can suppose that $\succ$ is not a weak order. For $x \in A$ and $k \in \mathbb{N}^*$, we note I_x^k the k-indifference set containing x (Cf. Section 4). Since $\succ^k \subset \succ^{k+1}$, we have $I_x^{k+1} \subset I_x^k$. Moreover, we have $\varphi_x \subset I_x^k$ for all $k \in \mathbb{N}^*$, and since $\gamma^{1/k}$ tends to the constant function $x \mapsto 1$ when k tends to $+\infty$, we have $\cap_k I_x^k = \varphi_x$. So if $x \sim y$, either $y \in \varphi_x$, either there exists $k \in \mathbb{N}^*$ such that $y \in I_x^k \backslash I_x^{k+1}$. □

6. Homothetic semiorders

In this section, we show that the biasing function is constant whenever the binary relation $\succ$ is a semiorder. In this manner, the representation of homothetic semiorders $\succ$ involves a ratio-scale and a constant multiplicative factor. First, we introduce the main result for homothetic semiorders:

Theorem 5. *Let A be a non-empty* $\mathbb{N}^*$*-set endowed with an Archimedean binary relation* $\succ$. *The three following conditions are equivalent* ($x, y \in A; m \in \mathbb{N}^*$):

(i) *There exists a function* $\varphi : A \rightarrow \mathbb{R}_+^*$ *and a number* $\alpha \in]0, 1]$ *such that* $\forall (x, y, m)$ *we have*

$$\begin{cases} x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y), \\ \varphi(mx) = m\varphi(x). \end{cases}$$

(ii) *The relation* $\succ$ *is a homothetic interval order such that* $\succ_1 = \succ_2$ (*in this case, we have* $\succ_0 = \succ_1 = \succ_2$).

(iii) *The relation* $\succ$ *is a homothetic semiorder.*

Moreover, if $\succ$ *is a homothetic semiorder, then the pair* (φ, α) *of* (i) *is unique up to multiplication of* φ *by a positive number.*

Proof. The implications (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (ii) are easy to verify.

Let us first show the implication (iii) $\Rightarrow$ (i). Suppose that $\succ$ is a homothetic semiorder. By Theorem 3, we can choose φ_1 and φ_2 such that $x \succ y \Leftrightarrow \varphi_1(x) > \varphi_2(y)$. We want to show that there exists $\lambda > 0$ such that $\varphi_2 = \lambda\varphi_1$. Suppose that there exist $y, t \in A$ such that $\frac{\varphi_2(y)}{\varphi_1(y)} \neq \frac{\varphi_2(t)}{\varphi_1(t)}$ and take $x, z \in A$. Without loss of generality, we can suppose that $\frac{\varphi_2(y)}{\varphi_1(y)} < \frac{\varphi_2(t)}{\varphi_1(t)}$. Hence, we have $\frac{\varphi_2(y)}{\varphi_2(t)} < \frac{\varphi_1(y)}{\varphi_1(t)}$. Up to replacing (y, t) by (my, nt) for some $m, n \in \mathbb{N}^*$, we can suppose that $\frac{\varphi_2(y)}{\varphi_2(t)} < 1 < \frac{\varphi_1(y)}{\varphi_1(t)}$.Hence, there exist $q, q' \in \mathbb{Q}_+^*$ such that

$$\begin{cases} \varphi_2(y) < q\varphi_1(x) \leqslant \varphi_2(t), \\ \varphi_1(t) \leqslant q'\varphi_2(z) < \varphi_1(y). \end{cases}$$

Let us write $q = \frac{a}{p}$ and $q' = \frac{b}{p}$ with $a, b, p \in \mathbb{N}^*$. Then, we have

$$\begin{cases} \varphi_2(py) < \varphi_1(ax) \leqslant \varphi_2(pt), \\ \varphi_1(pt) \leqslant \varphi_2(bz) < \varphi_1(py). \end{cases}$$

Therefore, we have

$$\begin{cases} ax \succ py \text{ and } py \succ bz, \\ ax \not\succ pt \text{ and } pt \not\succ bz, \end{cases}$$

which contradicts that $\succ$ is a semiorder. We hence have proven the implication (iii) $\Rightarrow$ (i).

To show the implication (ii) $\Rightarrow$ (i), suppose $\succ$ is a homothetic interval order such that $\succ_1 = \succ_2$. Choose an element $a \in A$. Then, from Corollary 1 and the uniqueness property of Theorem 1, there exists a unique constant $\beta > 0$ such that $r_{a,x} = \beta s_{a,x}$. By Corollary 1 and Theorem 3, this implies that $\succ_0 = \succ_1 = \succ_2$ and the function γ of Theorem 2 condition (i) is constant on A. $\square$

Let $\succ$ be an Archimedean homothetic semiorder. We choose a pair (φ, α) verifying condition (i) of Theorem 4. Then we have

$$y \in I_x \Leftrightarrow \alpha\varphi(x) \leqslant \varphi(y) \leqslant \alpha^{-1}\varphi(x).$$

This can also be written

$$x \sim y \Leftrightarrow \varphi(x) = \varphi(y)^{\times}_{\div}\alpha.$$

Hence, the upper and lower indifference thresholds do not depend on x: for all $x \in A$, we have $\delta_x^+ = \delta^+ = \alpha^{-1}$ and $\delta_x^- = \delta^- = \alpha$. This is illustrated in the following example.

Example 4. Consider the function $x = (x_1 X_1, x_2 X_2) \mapsto \varphi(x) = x_1^{2/5} x_2^{3/5}$ and define $\succ$ by $x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$ for all $x, y \in A$. Letting $\alpha = 0.9$, Fig. 7 shows the isocontours $\varphi^{10} = \{x : \varphi(x) = 10\}$ with $a, b \in \varphi^{10}$ and $\varphi^{20} = \{x : \varphi(x) = 20\}$ with $a', b' \in \varphi^{20}$ in bold lines. We depict the corresponding indifference sets I^{10} and I^{20} in plain lines. We have $\delta^+ = \frac{10}{9}$ and $\delta^- = \frac{9}{10}$.

Theorem 5 hence provides for a general formulation of Weber's law, which contends that the "just noticeable difference" maintains a constant ratio with respect to the intensity of the comparison stimulus. A constant multiplicative threshold of indifference (such as α in Theorem 5) is thus axiomatically derived from a semiorder. Since semiorders are traditionally interpreted as reflecting a constant additive threshold of indifference (see Pirlot & Vincke, 1997 for recent applications), we want now to discuss this issue in more details.

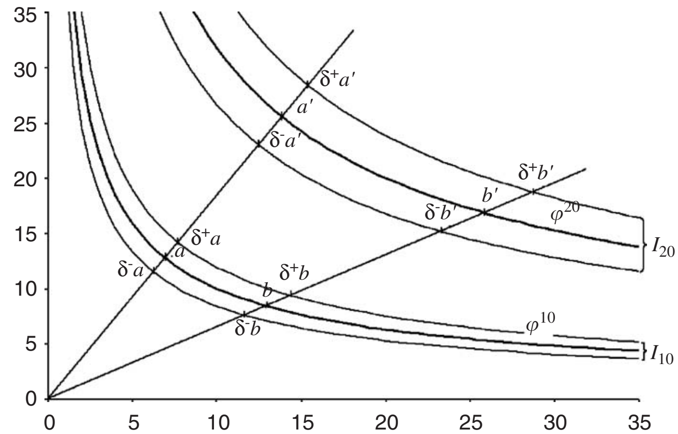

Fig. 7. Two indifference sets of a homothetic semiorder.

There is clearly a relation between a constant proportional threshold and a constant additive threshold. We can indeed reformulate the biased representation of Theorem 5 as a representation with a constant additive threshold by taking any logarithmic transformation of the representing function. For instance, if (φ, α) represents a homothetic semiorder $\succ$, then we have $x \succ y \Leftrightarrow \psi(x) > \psi(y) + \varepsilon$ and $\psi(mx) = \psi(x) + \log(m)$, where $\psi(x) = \log(\varphi(x))$ and $\varepsilon = -\log(\alpha) > 0$. In this manner, we have constructed a "derived scale" ψ such that $\varphi(x) > \varphi(y) \Leftrightarrow \psi(x) > \psi(y)$. This scale ψ allows the reformulation of the indifference relation as

$$x \sim y \Leftrightarrow \psi(x) = \psi(y) \pm \varepsilon$$

with $\varepsilon > 0$, which typically reflects error in measurement.

In terms of uniqueness, this formulation can be compared to the seminal result of Scott and Suppes (1958) who prove the existence of a function ψ over a finite set A such that $x \succ y \Leftrightarrow \psi(x) > \psi(y) + 1$. Their result cannot specify any uniqueness condition for the function ψ and thus, a fortiori, for the threshold of indifference. Hence, the function ψ cannot be interpreted as a measurement of the objects or stimuli and the additive threshold "1" does not meaningfully "measure" the interval between the value of objects (this "1" value is obtained by using the finiteness of A and could be replaced by any other value). To our knowledge, the same problem arises from other representations of finite semiorders (see the references given in introduction). With our theory of biased extensive measurement, both the objects and the threshold can be quantitatively measured. More precisely, the derived scale ψ is a *difference-scale* and the pair (ψ, ε) verifying $x \succ y \Leftrightarrow \psi(x) > \psi(y) + \varepsilon$ and $\psi(mx) = \psi(x) + \log(m)$ is unique up to the replacement of ψ by $\psi' = \psi + \mu$.

In terms of meaningfulness, the additive formulation ($x\sim y \Leftrightarrow \psi(x)=\psi(y)\pm\varepsilon$ with ψ being a difference-scale) shows both similar and distinct properties when compared to the multiplicative formulation ($x\sim y \Leftrightarrow \varphi(x)=\varphi(y)\overset{\times}{\div}\alpha$ with φ being a ratio-scale). Both ratio-scales and difference-scales allow meaningful comparisons of ratios and of differences (i.e. statements such as $\frac{f(x)}{f(y)}\geqslant\frac{f(x')}{f(y')}$ or $f(x)-f(y)\geqslant f(x')-f(y')$ are meaningful for f being a ratio-scale or a difference-scale). On the other hand, difference-scales allow meaningful numerical statements about the length of differences but not about the value of ratios, while ratio-scales allow meaningful statements about the value of ratios but not about the length of differences (i.e. statements such as "$\psi(x)-\psi(y)=2$" are meaningful for a difference scale but not for a ratio-scale, and statements such as "$\varphi(x)=2\varphi(y)$" are meaningful for a ratio-scale but not for a difference-scale). Finally, the expression $\varphi(mx)=m\varphi(x)$ in the multiplicative formulation becomes $\psi(mx)=\psi(x)+\log(m)$ in the additive formulation, which is not an algebraic property of the structure and seems to have little empirical meaning. In conclusion, the multiplicative formulation seems more "primitive" than the (derived) additive formulation but this may be a mere consequence of our axiomatization that builds on the property of homotheticity. Certainly, an axiomatization of an additive threshold of indifference such as $x\sim y\Leftrightarrow\varphi(x)=\varphi(y)\pm\varepsilon$, with φ being a *ratio-scale*, would clarify this issue, and we are not aware of such a result (for a discussion of themeaningfulness of the different types of statements related to uniqueness issues and to derived measurement, see Roberts, 1979, Sections 2.2, 2.3, 2.4, 2.5 and 6.1).

7. Homothetic semiorders with additive representation

Up to now, we have relied on a mere replication operation among objects to measure them with ratio-scales. When a concatenation operation among objects is available, it is natural to ask for the conditions under which we can construct a ratio-scale that is compatible with this operation, i.e. that verifies $\varphi(x\circ y)=\varphi(x)+\varphi(y)$. When this is the case, the measurement of an object formed by the concatenation of two objects is simply the sum of the measure for each object. In the presence of intransitive indifference, this does not mean that the independence condition must be verified. In the representation that we propose in this section, we show that a weaker condition is necessary and we term this condition pseudo-independence. We then show that semiorders that are pseudo-independent can be represented by an additive ratio-scale and a constant factor.

First, we need to introduce the structure of a *commutative semigroup*, that is a non-empty set A endowed with a map $A\times A\to A, (x,y)\mapsto x\circ y$ such that for all $x,y,z\in A$, we have $x\circ(y\circ z)=(x\circ y)\circ z$ (associativity) and $x\circ y=y\circ x$ (commutativity). Note that a commutative semigroup A is also a $\mathbb{N}^*$-set for the operation of $\mathbb{N}^*$ defined by $\mathbb{N}^*\times A\to A, (m,x)\mapsto mx=x\circ\cdots\circ x$ (m times). A real-valued function on a commutative semigroup A is then called *additive* if and only if, for all $x,y\in A$, $\varphi(x\circ y)=\varphi(x)+\varphi(y)$. Now, consider the following properties for a binary relation $\succ$ on A:

Independence: $\forall(x,y,z\in A)$ we have $x\succ y\Leftrightarrow x\circ z\succ y\circ z$;

Pseudo-independence: $\forall(x,y,z,t\in A)$ we have

$$\begin{cases}(x\succ y, z\succ t)\Rightarrow x\circ z\succ y\circ t,\\ (x\succsim y, z\succsim t)\Rightarrow x\circ z\succsim y\circ t.\end{cases}$$

When the relation $\succ$ is not a weak order, pseudo-independence is weaker than independence. For instance, the relation $\succ$ of Example 3 is pseudo-independent. However, it is independent if and only if $\lambda=\mu$. In this special case, the ratio-scale φ is additive. Note also that the relations $\succ$ of Examples 1, 2 and 4 are not independent nor pseudo-independent. We now prove the following theorem.

Theorem 6. *Let A be a commutative semigroup endowed with an Archimedean homothetic interval order $\succ$. The three following conditions are equivalent*:

(i) *There exists a function $\varphi: A\to\mathbb{R}_+^*$ and a number $\alpha\in]0,1]$ such that $\forall(x,y\in A)$ we have*

$$\begin{cases}x\succ y \;\Leftrightarrow\; \alpha\varphi(x)>\varphi(y),\\ \varphi(x\circ y)=\varphi(x)+\varphi(y).\end{cases}$$

(ii) *The relation $\succ_0$ is independent.*

(iii) *The relation $\succ$ is a pseudo-independent homothetic semiorder.*

Proof. The implication (i) $\Rightarrow$ (iii) is easy to verify and left to the reader.

We first prove (iii) $\Rightarrow$ (ii). Choose an element $a\in A$, and let $\varphi: A\to\mathbb{R}_+^*$ be the function defined by $\varphi(x)=r_{a,x}$. From Theorem 5, φ represents $\succ_0$ (as well as $\succ_1$ and $\succ_2$). Let $x,y\in A$. if $m,n,m',n'\in\mathbb{N}^*$ satisfy $ma\succsim nx$ and $m'a\succsim n'y$ then, since $\succ$ is pseudo-independent, we have $(nm'+n'm)a\succsim nn'(x\circ y)$. Therefore, we have $r_{a,x\circ y}\leqslant\frac{m}{n}+\frac{m'}{n'}$. Hence, we have $r_{a,x\circ y}\leqslant r_{a,x}+r_{a,y}$, i.e. $\varphi(x\circ y)\leqslant\varphi(x)+\varphi(y)$. In a similar manner, let $x,y\in A$. if $m,n,m',n'\in\mathbb{N}^*$ satisfy $mx\succ na$ and $m'y\succ n'a$ then $nm'(x\circ y)\succ(m'n+mn')a$. Hence, we have $s_{x\circ y,a}\leqslant\frac{mm'}{m'n+mn'}$. Therefore, $s_{x\circ y,a}\leqslant(\frac{n}{m}+\frac{n'}{m'})^{-1}$, i.e. $r_{a,x\circ y}\geqslant r_{a,x}+r_{a,y}$. We therefore have $\varphi(x\circ y)=\varphi(x)+\varphi(y)$ and the implication (iii) $\Rightarrow$ (ii) is proven.

To prove the implication (ii) $\Rightarrow$ (i), suppose that the relation $\succ_0$ is independent. Let $a\in A$. For $x,y,z\in A$, we have $x\circ z\succ_1 y\circ z \;\Leftrightarrow r_{a,x\circ z}>r_{a,y\circ z}$. We can replace a by $a\circ z\in A$ and we obtain

$$x\circ z\succ_1 y\circ z\Leftrightarrow r_{a\circ z,x\circ z}>r_{a\circ z,y\circ z}\Leftrightarrow r_{a,x}>r_{a,y}\Leftrightarrow x\succ_1 y\,.$$

Therefore, $\succ_1$ is independent. In the same way, we prove that $\succ_2$ is independent. Let $\varphi_0,\varphi_1,\varphi_2: A\to\mathbb{R}_+^*$ be the functions defined by $\varphi_1(x)=s_{a,a}r_{a,x}$, $\varphi_1(x)=s_{a,x}$ and

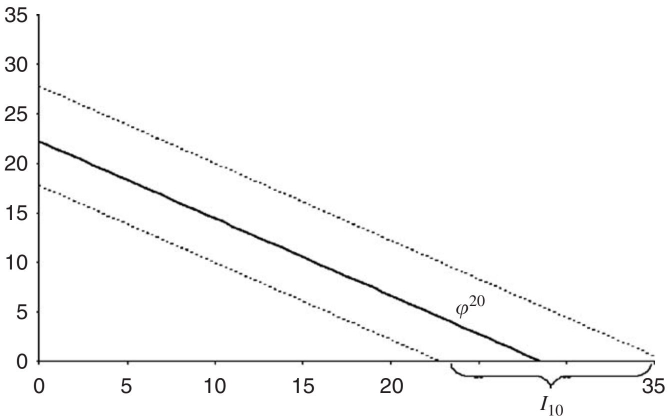

Fig. 8. One indifference set of an homothetic semiorder with additive representaion.

$\varphi_0(x) = (\varphi_1\varphi_2)^{1/2}$. From Theorem 3, for $i = 1, 2$, φ_i represents $\succ_i$, which are independent. Hence, for $x, y \in A$, we have

$$\begin{aligned}\varphi_0(x \circ y)^2 &= \varphi_0(x)^2 + \varphi_0(y)^2 + \varphi_1(x)\varphi_2(y) + \varphi_1(y)\varphi_2(x) \\ &= [\varphi_0(x) + \varphi_0(y)]^2 + ([\varphi_1(x)\varphi_2(y)]^{1/2} \\ &\quad - [\varphi_1(y)\varphi_2(x)]^{1/2})^2\end{aligned}$$

from which we deduce that $[\varphi_1(x)\varphi_2(y)]^{1/2} = [\varphi_1(y)\varphi_2(x)]^{1/2}$, i.e. that $\varphi_2 = \lambda\varphi_1$ for some $\lambda > 0$. Therefore, $\succ$ is a semiorder and the implication (ii) $\Rightarrow$ (i) is verified. □

We illustrate this result with a two-dimensional space $A = \mathbb{R}_+^* X_1 \times \mathbb{R}_+^* X_2$ endowed with the operation $\circ$ defined by $(x_1X_1, x_2X_2) \circ (x_1'X_1, x_2'X_2) = ((x_1 + x_1')X_1, (x_2 + x_2')X_2)$ with $x_1, x_2 \in \mathbb{R}_+^*$.

Example 5. Consider the function $x = (x_1X_1, x_2X_2) \mapsto \varphi(x) = \lambda x_1 + \mu x_2$ and define $\succ$ by $x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$ for all $x, y \in A$. Letting $\alpha = 0.8$, $\lambda = 0.7$ and $\mu = 0.9$, Fig. 8 shows the isocontours $\varphi^{20} = \{x : \varphi(x) = 20\}$ in bold, with the corresponding indifference set I^{20} delimited by dotted straight lines.

8. Concluding remarks

With theories of biased extensive measurement, we extend extensive measurement to phenomena where the measurement process is distorted or biased, leading to a lack of discrimination and a lack of consistency. We show that a fully quantitative measurement of objects is possible even if transitivity of indifference and independence are violated. Moreover, we characterize the extent to which the measurement process is distorted with a biasing function. Such a bias is unique and does not depend on the quantity of objects or intensity of the stimuli. We build on this biasing function to propose new concepts that allow for the precise measurement of thresholds of indifference. In this manner, biased extensive measurement combines the measurement of objects with the measurement of the measuring process.

Our approach suggests to extend ratio-scale measurement and extensive measurement to a broader class of phenomena, in particular in psychological sciences. The observation of insensitivity and/or inconsistency shall not necessarily be interpreted as a nuisance for the measurement of objects or stimuli but as a source of information about the underlying process, for instance the psychological processes. For example, this should be relevant for the mathematical foundations of preferences, where the reduction of preferences to a measure of each object (its "utility") has proven to be empirically suspicious (see, e.g., Slovic, 1995). Indeed, the biasing function could help to explain the part of preferences that lies beyond the value of each object. The analogy with the biased balance suggests that this part of preferences resides in the subject who is acting as a measuring device.

Finally, we would like to note that, in the case of homogeneous sets, we have been able to considerably extend the results of biased extensive measurement to any binary relation that is positive and homothetic (Axioms 1 and 4 above). Values greater than 1 are then possible for the biasing function which, in the homogeneous case, remains constant and unique (see Le Menestrel & Lemaire, 2006). We are working on generalizing these results to the non-homogeneous case.

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References

Abbas, M. (1995). Any complete preference structure without circuit admits an interval representation. *Theory and Decision*, *39*, 115–126.

Bosi, G. (2002). Semicontinuous representability of homothetic interval orders by means of two homogeneous functionals. *Theory and Decision*, *52*, 303–312.

Bosi, G., Candeal, J. C., Induráin, E., & Zudaire, M. (2005). Existence of homogeneous representations of interval orders on a cone in a topological vector space. *Social Choice Welfare*, *45*, 45–61.

Bridges, D. S. (1986). Numerical representation of interval orders on a topological space. *Journal of Economic Theory*, *38*, 160–166.

Candeal, J. C., De Miguel, J. R., & Indurain, E. (1997). Topological additively representable semigroups. *Journal of Mathematical Analysis and Applications*, *210*, 375–389.

Candeal, J. C., & Indurain, E. (1995). Homothetic and weakly homothetic preferences. *Journal of Mathematical Economics*, *24*, 147–158.

Chateauneuf, A. (1987). Continuous representation of a preference relation on a connected topological space. *Journal of Mathematical Economics*, *16*, 139–146.

De Miguel, J. R., Candeal, J. C., & Indurain, E. (1996). Archimedeaness and Additive Utility on Totally Ordered Semigroups. *Semigroup Forum*, *52*, 335–347.

Doignon, J. P., Ducamp, A., & Falmagne, J. C. (1984). On realizable biorders and the biorder dimension of a relation. *Journal of Mathematical Psychology*, *28*, 73–109.

Fishburn, P. C. (1973). Interval representations for interval orders and semiorders. *Journal of Mathematical Psychology*, *10*, 91–105.

Fishburn, P. C. (1985). *Interval orders and interval graphs*. New York: Wiley.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement* (Vol. I). San Diego: Academic Press.

Lemaire, B., & Le Menestrel, M. (2006). Homothetic interval orders. *Discrete Mathematics*, *306*, 1669–1683.

Le Menestrel, M., & Lemaire, B. (2004). Biased extensive measurement: The homogeneous case. *Journal of Mathematical Psychology*, *48*, 9–14.

Le Menestrel, M., & Lemaire, B. (2006). Ratio-scale measurement with intransitivity or incompleteness: The homogeneous case. *Theory and Decision*, *60*, 207–217.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, *24*, 178–191.

Oloriz, E., Candeal, J. C., & Indurain, E. (1998). Representability of Interval Orders. *Journal of Economic Theory*, *78*, 219–227.

Pirlot, M., & Vincke, Ph. (1997). *Semiorders*. Dordrecht: Kluwer Academic Publishers.

Poincaré, H., 1903. L'Espace et ses trois dimensions. *Revue de métaphysique et de morale*, II (pp. 281–301 and pp. 407–429). Rep. in Poincaré, H. (2001). *The value of science : Essential writings of Henri Poincaré*. New York: Modern Library.

Roberts, F. S. (1979). *Measurement Theory with applications to decision making, utility and the social sciences*. Reading: Addison-Wesley.

Scott, D., & Suppes, P. (1958). Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, *23*, 113–128.

Slovic, P. (1995). The construction of preferences. *American Psychologist*, *50*, 271–364.

Stevens, S. (1946). On the theory of scales of measurement. *Science*, *103*, 122–136.

Suppes, P., Krantz, D. H., Luce, R. D., & Tversky, A. (1989). *Foundations of measurement* (Vol. II). San Diego: Academic Press.