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Generalized homothetic biorders☆

Bertrand Lemaire [a], Marc Le Menestrel [b,∗]

[a] Institut de Mathématiques de Luminy et UMR 6206 du CNRS, Université Aix-Marseille II, Case Postale 907, 163 Avenue de Luminy, 13288 Marseille Cedex 9, France
[b] Universitat Pompeu Fabra, Departament d'Economia i Empresa, Ramon Trias Fargas 25-27, 08005-Barcelona, Spain

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ABSTRACT

In this paper, we study the binary relations R on a nonempty $\mathbb{N}^*$-set A which are *h-independent* and *h-positive* (cf. the introduction below). They are called *homothetic positive orders*. Denote by $\mathbb{B}$ the set of intervals of $\mathbb{R}$ having the form $[r, +\infty[$ with $0 < r \leq +\infty$ or $]q, \infty[$ with $q \in \mathbb{Q}_{\geq 0}$. It is a $\mathbb{Q}_{>0}$-set endowed with a binary relation $>$ extending the usual one on $\mathbb{R}_{>0}$ (identified with a subset of $\mathbb{B}$ via the map $r \mapsto [r, +\infty[$). We first prove that there exists a *unique* map $\Phi_R : A \times A \to \mathbb{B}$ such that (for all $x, y \in A$ and all $m, n \in \mathbb{N}^*$) we have $\Phi(mx, ny) = mn^{-1} \cdot \Phi(x, y)$ and $x\,R\,y \Leftrightarrow \Phi_R(x, y) > 1$. Then we give a characterization of the homothetic positive orders R on A such that there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{B}$ satisfying $x\,R\,y \Leftrightarrow u_1(x) > u_2(y)$. They are called *generalized homothetic biorders*. Moreover, if we impose some natural conditions on the sets $u_1(A)$ and $u_2(A)$, the representation (u_1, u_2) is "uniquely" determined by R. For a generalized homothetic biorder R on A, the binary relation R_1 on A defined by $x\,R_1\,y \Leftrightarrow \Phi_R(x, y) > \Phi_R(y, x)$ is a *generalized homothetic weak order*; i.e. there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{B}$ such that (for all $x, y \in A$) we have $x\,R_1\,y \Leftrightarrow u(x) > u(y)$. As we did in [B. Lemaire, M. Le Menestrel, Homothetic interval orders, Discrete Math. 306 (2006) 1669–1683] for homothetic interval orders, we also write "the" representation (u_1, u_2) of R in terms of u and a twisting factor.

This paper proposes a generalization of [13] in which we had studied *homothetic interval orders* on a nonempty $\mathbb{N}^*$-set A. Let us recall that such an order R is a nonempty binary relation, *h-independent* in the sense that $x\,R\,y \Leftrightarrow mx\,R\,my$ for all $x, y \in A$ and all $m \in \mathbb{N}^*$, and satisfying a series of properties that ensure the existence of two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}_{>0}$ such that $x\,R\,y \Leftrightarrow u_1(x) > u_2(y)$ with $u_1 \leq u_2$. Moreover, the pair (u_1, u_2) is *unique* up to multiplication by a positive scalar. Besides h-independence, the most striking properties of homothetic interval orders are:

- *asymmetry*: $x\,R\,y \Rightarrow y\,(-R)\,x$ where $-R$ means the negation of R;
- *h-positivity*: for all $m, n \in \mathbb{N}^*$ such that $m > n$, we have $x\,R\,y \Rightarrow mx\,R\,ny$;
- *h-super-Archimedean*[1]: if $x\,R\,y$, then there exists $m \in \mathbb{N}^*$ such that $mx\,R\,(m+1)y$.

Note that asymmetry implies

- *irreflexivity*: $x\,(-R)\,x$.

Of all these properties, this paper first retains only two: h-independence and h-positivity.

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∗ Corresponding author.
E-mail addresses: lemaire@iml.univ-mrs.fr (B. Lemaire), marc.lemenestrel@upf.edu (M. Le Menestrel).

[1] In [13], we called this property h-Archimedean but the terminology of the present paper is more in line with the literature (see e.g. [6]).

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Let $\mathcal{R}(A)$ be the set of binary relations on A that are h-independent and h-positive. Abandoning the h-super-Archimedean property naturally leads to enrich the range of the representation: denote by $\mathbb{R}^\natural$ the set of intervals of $\mathbb{R}$ having the form $[r, +\infty[$ or $]r, +\infty[$ with $-\infty < r \leq +\infty$, endowed with the order $\leq$ inverse of the one given by inclusion. We identify $r \in \mathbb{R}$ with the closed interval $[r, +\infty[$, and we note r^+ the open interval $]r, +\infty[$. We denote by ∞ the empty interval $]+\infty, +\infty[$. In this manner, the relation $\leq$ on $\mathbb{R}^\natural$ extends the relation on $\mathbb{R}$, and we denote by $>$ its negation: for two intervals I, I' in $\mathbb{R}^\natural$, we then have $I > I' \Leftrightarrow I' \not\subset I$. We also endow $\mathbb{R}^\natural$ with a $\mathbb{R}_{>0}$-set structure extending the one of $\mathbb{R}$ (cf. Section 1). Finally, we define:

$$\mathbb{B} = \mathbb{R}_{>0} \cup \{q^+ : q \in \mathbb{Q}_{>0}\} \cup \{0^+, \infty\},$$
$$\mathbb{A} = \mathbb{R}_{>0} \cup \{0^+, \infty\}.$$

We have the inclusions $\mathbb{A} \subset \mathbb{B} \subset \mathbb{R}^\natural$; and $\mathbb{A}$ and $\mathbb{B}$ are respectively a sub-$\mathbb{R}_{>0}$-set and a sub-$\mathbb{Q}_{>0}$-set of $\mathbb{R}^\natural$.

Our first result is the following (4.2): *for all relation $R \in \mathcal{R}(A)$, there exists a* unique *function $\Phi_R : A \times A \to \mathbb{B}$ satisfying* (*for all $x, y \in A$ and all $m, n \in \mathbb{N}^*$*):

(1) $\Phi_R(mx, ny) = \frac{m}{n} \cdot \Phi_R(x, y)$;
(2) $x\,R\,y \Leftrightarrow \Phi_R(x, y) > 1$.

Conversely, any binary relation R on A such that there exists a function $\Phi : A \times A \to \mathbb{B}$ satisfying (1) and (2) belongs to $\mathcal{R}(A)$.

Denote by $\mathcal{R}'(A)$ the subset of $\mathcal{R}(A)$ consisting of h-super-Archimedean relations. We verify that for $R \in \mathcal{R}(A)$, we have $R \in \mathcal{R}'(A)$ if and only if $\Phi_R(A \times A) \subset \mathbb{A}$.

We then introduce a notion that extends homothetic interval orders: a relation $R \in \mathcal{R}(A)$ is said to be *a generalized homothetic biorder* if there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}^\natural_{>0}$ such that $x\,R\,y \Leftrightarrow u_1(x) > u_2(y)$; in which case we say that the pair (u_1, u_2) *represents* R. We show that if $R \in \mathcal{R}(A)$ is a generalized homothetic biorder, then the pair (u_1, u_2) which represents R is *unique* up to multiplication by a positive scalar (6.5). In fact, the correct formulation of this uniqueness property is slightly more complicated (cf. Section 6), and leads us to distinguish three cases: $\Phi_R(A \times A) \subset \{0^+, \infty\}$; $\Phi_R(A \times A) \subset \mathbb{A} \setminus \{0^+, \infty\}$; and $\Phi_R(A \times A) \subset \mathbb{B} \setminus \mathbb{A}$. In particular, if R is a h-super-Archimedean generalized homothetic biorder on A such that $\Phi_R(A \times A) \not\subset \{0^+, \infty\}$, then there exists a "unique" representation (u_1, u_2) of R *in* $\mathbb{A}$, i.e. such that $u_i(A) \subset \mathbb{A}\,(i = 1, 2)$.

Before going on, let us consider the three following properties for a binary relation R on A:

- *Ferrers*: xRy and $zRt \Rightarrow xRt$ or zRy;
- *negative transitivity*: $x\,(-R)\,y\,(-R)\,z \Rightarrow x\,(-R)\,z$.

Recall that R is called:

- a *biorder* if it is Ferrers;
- an *interval order* if is irreflexive and Ferrers;
- a *weak order* if is asymmetric and negatively transitive.

So we have the implications:

weak order $\Rightarrow$ interval order $\Rightarrow$ biorder.

Usually, a biorder R on A is said to be *representable* if there exist two functions $u_1, u_2 : A \to \mathbb{R}_{>0}$ such that

$$x\,R\,y \Leftrightarrow u_1(x) > u_2(y).$$

Note that if $u_1 \leq u_2$ (resp. $u_1 = u_2$), then R is an interval order (resp. a weak order).

Thus homothetic interval orders are particular cases of representable interval order. And the notion of generalized homothetic biorder is a twofold extension: firstly we enlarge the space of the representation, in the sense that the two functions u_1 and u_2 on A may take their values in $\mathbb{R}^\natural_{>0}$ instead of $\mathbb{R}_{>0}$; secondly we remove the condition $u_1 \leq u_2$. A generalized homothetic biorder represented by a pair (u_1, u_2) such that $u_1 \leq u_2$ (resp. $u_1 = u_2$) is called a *generalized homothetic interval order* (resp. a *generalized homothetic weak order*).

Let us return to the contents of the paper. We then study the characterization of the subset $\mathcal{R}_\bullet(A) \subset \mathcal{R}(A)$ of generalized homothetic biorders. Note that in [12], we have studied the case where A is homogeneous (i.e. such that for all $x, y \in A$, there exist $m, n \in \mathbb{N}^*$ such that $mx = ny$). In that special case, it is easy to show that any positive homothetic order is also a homothetic biorder, i.e. we have $\mathcal{R}_\bullet(A) = \mathcal{R}(A)$. But this equality is no longer true in general. In Section 9, we identify a (finite!) set of properties that characterize $\mathcal{R}_\bullet(A)$. These properties are in fact compatibility properties between R and its dual relation $R^\vee \in \mathcal{R}(A)$, defined by

$$x\,R^\vee\,y \Leftrightarrow y\,(-R)\,x.$$

We show that if $R \in \mathcal{R}_\bullet(A)$, then the binary relation R_1 on A given by

$$x\,R_1\,y \Leftrightarrow \Phi_R(x, y) > \Phi_R(y, x)$$

is a generalized homothetic weak order (11.1). This allows us to extend the representation of a homothetic interval order introduced in [11,13] to generalized homothetic biorders (11.3): for $R \in \mathcal{R}_\bullet(A)$, there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}^\natural_{>0}$ and a map $\gamma : A/\mathbb{N}^* \to \mathbb{A}$, such that (cf. the writing conventions in Section 11)

$$x\,R\,y \Leftrightarrow \gamma(x)\cdot u(x) \Leftrightarrow \gamma(y)^{-1}\cdot \tilde{u}(y)$$

with

$$\tilde{u}(y) = \begin{cases} u(y) & \text{if } u(y) \in \mathbb{A} \\ r & \text{if } u(y) = r^+. \end{cases}$$

Moreover, if we ask the pair (u, γ) to satisfy some natural conditions, then it is unique up to replacing it by $(\lambda \cdot u, \gamma)$ for a $\lambda \in \mathbb{R}_{>0}$.

Let us conclude this introduction with some remarks about the nature of our results, and their link with the literature on the topic. Our algebraic study of homothetic orders began with homothetic semiorders on homogeneous sets in [11] and was later generalized to homothetic interval orders and homothetic semiorders on general sets in [13]. As we said, we extended the homogeneous case to positive orders in [12]. Following the work of Ducamp and Falmagne [8], the term of biorder has been introduced by Doignon et al. [7] who identify conditions for their representation by two functions. In their terminology, the domains of the two functions are not necessarily identical but Aleskerov and Masatlioglu [3] use the same terminology for the particular case of a single domain, like we do in this paper. The same definition for biorder is also used in the useful survey of threshold representations by Aleskerov, Bouyssou and Monjardet [2]. Recent papers such as Bosi et al. [4] and [5] propose a (semi)continuous representation of interval orders and state that it can be extended to biorders. Compared with these "ordinal" approaches, the originality of our work resides in its algebraic nature, which allows us to disregard the consideration of a topology on the set A. Moreover, we provide uniqueness properties that allow us to "measure" the intervals or thresholds of our representations. As for the set of open and closed intervals (possibly empty) of the real numbers to represent possibly non-super-Archimedean orders, it has been used by Nakamura [14]. Another possibility is Narens [15], where non-standard models of the real numbers are considered to treat the abandon of the super-Archimedean condition. We would also like to point out a recent example of a structure without transitivity but with asymmetry in Abbas et al. [1] (note their structures are not necessarily representable by two functions like in this paper). Finally, a useful review of orders that are asymmetric and transitive is Fishburn [10] and a review of nontransitive (but asymmetric) representations can be found in Fishburn [9].

Notations/writing conventions. We denote by $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ the sets of real numbers, rational numbers and integers; and we write $\mathbb{N}^* = \mathbb{Z}_{>0}$. If X and Y are two subsets of $\mathbb{R}$, we write $XY = \{rs : r \in X,\ s \in Y\}$.

If R and R' are two binary relations on a set A, for $x, y, z \in A$, we write $x\,R\,y\,R'\,z \Leftrightarrow x\,R\,y$ and $y\,R'\,z$.

The symbol $\coprod$ means disjoint union.

1. The sets $\mathbb{R}^\natural_{>0}$, $\mathbb{B}$ and $\mathbb{A}$

Recalling the definition of $\mathbb{R}^\natural$ given in the introduction, for two intervals I, I' in $\mathbb{R}^\natural$, we have $I \le I' \Leftrightarrow I \supset I'$. Hence, the relation $\le$ is a total order on $\mathbb{R}^\natural$ and $\le$ on $\mathbb{R}^\natural$ extends $\le$ on $\mathbb{R}$: it is given by ($r, s \in \mathbb{R};\ t \in \mathbb{R}^\natural$):

$$r^+ \le s^+ \Leftrightarrow r \le s^+ \Leftrightarrow r \le s,$$
$$r^+ \le s \Leftrightarrow r < s,$$
$$t \le \infty.$$

For $r, s \in \mathbb{R}^\natural$, we define

$$r \ge s \Leftrightarrow s \le r,$$
$$r < s \Leftrightarrow \{r \le s \text{ and } r \ne s\} \Leftrightarrow s > r.$$

We also endow $\mathbb{R}^\natural$ with the structure of an additive monoid extending the one of $\mathbb{R}$, defined by ($r, s \in \mathbb{R}$):

$$r + s^+ = r^+ + s^+ = (r+s)^+,$$
$$r + \infty = r^+ + \infty = \infty + \infty = \infty.$$

Notice that the relation $\le$ on $\mathbb{R}^\natural$ is compatible with the operation $+$. In this manner, $(\mathbb{R}, +, \le)$ is an ordered additive sub-monoid of $(\mathbb{R}^\natural, +, \le)$.

Let $\mathbb{R}^\natural_{>0} = \{r \in \mathbb{R}^\natural : r > 0\}$; this is a sub-semigroup of $\mathbb{R}^\natural$. Consider $\mathbb{R}^\natural_{>0} \to \mathbb{R}^\natural_{>0},\ x \mapsto x^\vee$ the map defined by ($r \in \mathbb{R}_{>0}$):

$$r^\vee = (r^{-1})^+, \qquad (r^+)^\vee = r^{-1},$$
$$(0^+)^\vee = \infty, \qquad \infty^\vee = 0^+.$$

It is an involution: for $r \in \mathbb{R}^\natural_{>0}$, we have $(r^\vee)^\vee = r$. In particular, it is a bijective map. And for $r, s \in \mathbb{R}^\natural_{>0}$, we have

$$r \le s \Leftrightarrow r^\vee \ge s^\vee.$$

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Let $\mathbb{R}_{>0} \times \mathbb{R}^{\natural}_{>0} \to \mathbb{R}^{\natural}_{>0}$, $(s, r) \mapsto s \cdot r$ be the $\mathbb{R}_{>0}$-set structure on $\mathbb{R}^{\natural}_{>0}$ defined by ($s,\ r \in \mathbb{R}_{>0}$):

$$s \cdot r = sr, \qquad s \cdot r^+ = (sr)^+,$$
$$s \cdot 0^+ = 0^+, \qquad s \cdot \infty = \infty.$$

For $(s, r) \in \mathbb{R}_{>0} \times \mathbb{R}^{\natural}_{>0}$, we have

$$(s \cdot r)^\vee = s^{-1} \cdot r^\vee.$$

Let

$$\mathbb{B} = \mathbb{R}_{>0} \cup \{q^+ : q \in \mathbb{Q}_{>0}\} \cup \{0^+, \infty\},$$
$$\mathbb{A} = \mathbb{R}_{>0} \cup \{0^+, \infty\},$$
$$\mathbb{Q}^{\natural}_{>0} = \mathbb{Q}_{>0} \cup \{q^+ : q \in \mathbb{Q}_{>0}\} \cup \{0^+, \infty\}.$$

We have the inclusions $\mathbb{A} \subset \mathbb{B}$ and $\mathbb{Q}^{\natural}_{>0} \subset \mathbb{B}$. Moreover, we have $\mathbb{R}_{>0} \cdot \mathbb{A} = \mathbb{A}$, $\mathbb{Q}_{>0} \cdot \mathbb{B} = \mathbb{B}$ and $\mathbb{Q}_{>0} \cdot \mathbb{Q}^{\natural}_{>0} = \mathbb{Q}^{\natural}_{>0}$; i.e. $\mathbb{A}$ is a sub-$\mathbb{R}_{>0}$-set of $\mathbb{R}_{>0}$, and $\mathbb{B}$ and $\mathbb{Q}^{\natural}_{>0}$ are sub-$\mathbb{Q}_{>0}$-sets of $\mathbb{R}^{\natural}_{>0}$. The involution $x \mapsto x^\vee$ induces by restriction three bijective maps

$$\mathbb{B} \to \mathbb{B}^\vee = \{r^+ : r \in \mathbb{R}_{\geq 0}\} \cup \mathbb{Q}_{>0} \cup \{\infty\},$$
$$\mathbb{A} \to \mathbb{A}^\vee = \{r^+ : r \in \mathbb{R}_{\geq 0}\} \cup \{\infty\},$$
$$\mathbb{Q}^{\natural}_{>0} \to \mathbb{Q}^{\natural}_{>0}.$$

And we have

$$\mathbb{B} \cup \mathbb{B}^\vee = \mathbb{A} \cup \mathbb{A}^\vee = \mathbb{R}^{\natural}_{>0},$$
$$\mathbb{B} \cap \mathbb{B}^\vee = \mathbb{A} \cap \mathbb{A}^\vee = \mathbb{Q}^{\natural}_{>0}.$$

Remark 1.1. Let $r,\ s \in \mathbb{B}$ such that $r > s$. Then $r \neq 0^+$, $s \neq \infty$, and either $r \neq s^+$ and then there exists a $q \in \mathbb{Q}_{>0}$ such that $r > q > s$; or $r = s^+$ and then $s \in \mathbb{Q}_{>0}$. So in both cases, there exists a $q \in \mathbb{Q}_{>0}$ such that $r > q \geq s$. ⋆

2. The sets $\mathcal{R}(A)$, $\mathcal{R}'(A)$ and $\mathcal{R}''(A)$

Recall the definition of $\mathcal{R}(A)$ and $\mathcal{R}'(A)$ given in the introduction. Denote by $R_\emptyset$ and R_∞ the binary relations on A that are respectively *empty* and *trivial*; i.e. for all $x,\ y \in A$, we have $x(-R_\emptyset)y$ and $x\,R_\infty\,y$. Both belong to $\mathcal{R}'(A)$. And we have $R_\emptyset^\vee = R_\infty$ and $R_\infty^\vee = R_\emptyset$.

The set $\mathcal{R}(A)$ is endowed with a structure of a $\mathbb{Q}_{>0}$-set: for $R \in \mathcal{R}(A)$ and $q \in \mathbb{Q}_{>0}$, we write $q = \frac{m}{n}$ with $m,\ n \in \mathbb{N}^*$, and we denote R^q the binary relation on A defined by $x\,R^q\,y \Leftrightarrow mx\,R\,ny$. By $({}_\mathrm{h}\mathrm{I})$, the relation R^q is well-defined, i.e. it does not depend on the choice of m and n such that $q = \frac{m}{n}$. It clearly belongs to $\mathcal{R}(A)$. For $R \in \mathcal{R}(A)$ and $q,\ q' \in \mathbb{Q}_{>0}$, we have

$$(R^q)^{q'} = R^{qq'}.$$

And the subset $\mathcal{R}'(A)$ of $\mathcal{R}(A)$ is $\mathbb{Q}_{>0}$-stable.

For $R \in \mathcal{R}(A)$, we denote R' the binary relation on A defined by

$$x\,R'\,y \Leftrightarrow (\text{for all } m,\ n \in \mathbb{N}^* \text{ such that } m > n, \text{ we have } mx\,R\,ny).$$

Then $R' \in \mathcal{R}'(A)$, and the map $\mathcal{R}(A) \to \mathcal{R}'(A),\ R \mapsto R'$ is the identity relation on $\mathcal{R}'(A)$.

For $R \in \mathcal{R}(A)$, the relation $R^\vee$ still belongs to $\mathcal{R}(A)$ (easy to check). And the map $\mathcal{R}(A) \to \mathcal{R}(A),\ R \mapsto R^\vee$ is an involution: for $R \in \mathcal{R}(A)$, we have $(R^\vee)^\vee = R$. For $R \in \mathcal{R}(A)$ and $q \in \mathbb{Q}_{>0}$, we have

$$(R^q)^\vee = (R^\vee)^{q^{-1}},$$
$$(R^q)' = (R')^q.$$

Let $\mathcal{R}'(A)^\vee$ be the subset of $\mathcal{R}(A)$ formed by relations satisfying the following condition $({}_\mathrm{h}\mathrm{A})^\vee$ (for all $x,\ y \in A$):

$({}_\mathrm{h}\mathrm{A})^\vee$: if $(m+1)x\,R\,my$ for all $m \in \mathbb{N}^*$, then $x\,R\,y$.

The involution $\mathcal{R}(A) \to \mathcal{R}(A),\ R \mapsto R^\vee$ induces by restriction two bijective maps, that are the inverse of one another:

$$\mathcal{R}'(A) \to \mathcal{R}'(A)^\vee,$$
$$\mathcal{R}'(A)^\vee \to \mathcal{R}'(A).$$

Let $\mathcal{R}''(A) = \mathcal{R}'(A) \cap \mathcal{R}'(A)^\vee$. The involution $\mathcal{R}(A) \to \mathcal{R}(A),\ R \mapsto R^\vee$ induces by restriction a bijective map

$$\mathcal{R}''(A) \to \mathcal{R}''(A).$$

Notice that since $R_\emptyset,\ R_\infty \in \mathcal{R}'(A)$ and $R_\infty = R_\emptyset^\vee$, we have $R_\emptyset,\ R_\infty \in \mathcal{R}''(A)$.

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3. The invariants $\mathcal{P}^R_{x,y}$, $s^R_{x,y}$ and $t^R_{x,y}$

Let $R \in \mathcal{R}(A)$, and let $x,\ y \in A$. Let

$$\mathcal{P}^R_{x,y} = \{mn^{-1} : m,\ n \in \mathbb{N}^*,\ mx\,R\,ny\} \subset \mathbb{Q}_{>0}.$$

If $q \in \mathcal{P}^R_{x,y}$, then we have the inclusion $\mathbb{Q}_{\geq q} \subset \mathcal{P}^R_{x,y}$. Therefore

$$s^R_{x,y} = \bigcup_{q \in \mathcal{P}^R_{x,y}} [q, +\infty[$$

is an element of $\mathbb{R}^{\natural}_{>0}$, and we have $s^R_{x,y} \cap \mathbb{Q} = \mathcal{P}^R_{x,y}$. We distinguish two cases: either for all $q \in \mathcal{P}^R_{x,y}$, we have $\mathcal{P}^R_{x,y} \cap \mathbb{Q}_{<q} \neq \emptyset$, and then

$$s^R_{x,y} = \begin{cases} [\inf_{\mathbb{R}}(\mathcal{P}^R_{x,y})]^+ & \text{if } \mathcal{P}^R_{x,y} \neq \emptyset \\ \infty & \text{if not;} \end{cases}$$

or there exists a $s \in \mathbb{Q}_{>0}$ such that $\mathcal{P}^R_{x,y} = \mathbb{Q}_{\geq s}$, and then $s^R_{x,y} = s$. In particular, we have $s^R_{x,y} \in \mathbb{B}^{\vee}$. The triplet (x, R, y) is said to be *super-Archimedean* in the first case, and *non super-Archimedean* in the second case. Notice that $R \in \mathcal{R}'(A)$ if and only if for all $x',\ y' \in A$, the triplet (x', R, y') is super-Archimedean.

Notation 3.1. For $R \in \mathcal{R}(A)$, we note $\mathcal{A}^R$ the set of $(x, y) \in A \times A$ such that the triplet (x, R, y) is super-Archimedean, and we let $\mathcal{B}^R = (A \times A) \setminus \mathcal{A}^R$. We also define

$$\mathcal{A}^R_1 = \{x \in A : (x, y) \in \mathcal{A}^R,\ \forall y \in A\},$$
$$\mathcal{A}^R_2 = \{y \in A : (x, y) \in \mathcal{A}^R,\ \forall x \in A\},$$

and

$$\mathcal{B}^R_i = A \setminus \mathcal{A}^R_i \quad (i = 1,\ 2).$$

For $R \in \mathcal{R}(A)$, since $\mathcal{B}^R \subset \mathcal{B}^R_1 \times \mathcal{B}^R_2$, we have the decomposition

$$\mathcal{A}^R = \mathcal{A}^R \cap (\mathcal{B}^R_1 \times \mathcal{B}^R_2) \coprod \mathcal{A}^R \cap (\mathcal{B}^R_1 \times \mathcal{A}^R_2) \coprod \mathcal{A}^R \cap (\mathcal{A}^R_1 \times \mathcal{B}^R_2) \coprod \mathcal{A}^R_1 \times \mathcal{A}^R_2. \tag{3.2}$$

Let $R \in \mathcal{R}(A)$, and let $x,\ y \in A$. Let

$$t^R_{x,y} = (s^R_{x,y})^{\vee} \in \mathbb{B}.$$

From what precedes, we have

$$(x, R, y) \in \mathcal{A}^R \Leftrightarrow t^R_{x,y} \in \mathbb{A}.$$

Moreover, we have

$$x\,R\,y \Leftrightarrow 1 \in \mathcal{P}^R_{x,y} \Leftrightarrow s^R_{x,y} \leq 1 \Leftrightarrow t^R_{x,y} \geq 1^+ \Leftrightarrow t^R_{x,y} > 1. \tag{3.3}$$

And for all $m,\ n \in \mathbb{N}^*$, we have

$$t^R_{mx,ny} = \frac{m}{n} \cdot t^R_{x,y}.$$

Lemma 3.4. *Let $R \in \mathcal{R}(A)$.*

(1) *For $(x, y) \in \mathcal{A}^R$, we have*

$$t^{R^{\vee}}_{y,x} = \begin{cases} r & \text{if } s^R_{x,y} = r^+ \text{ with } r \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0} \\ s^R_{x,y} & \text{if } s^R_{x,y} \in \{0^+, \infty\} \text{ or if } s^R_{x,y} = q^+ \text{ with } q \in \mathbb{Q}_{>0}. \end{cases}$$

(2) *For $(x, y) \in \mathcal{B}^R$, we have*

$$t^{R^{\vee}}_{y,x} = s^R_{x,y} \in \mathbb{Q}_{>0}.$$

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Proof. Let $(x, y) \in A \times A$. We have $\mathcal{P}^{R^\vee}_{y,x} = \{q^{-1} : q \in \mathbb{Q}_{>0} \setminus \mathcal{P}^R_{x,y}\}$.

Suppose first that $t^R_{x,y} \in \{0^+, \infty\}$. Then $(y, x) \in \mathcal{A}^{R^\vee}$ and

$$t^{R^\vee}_{y,x} = s^R_{x,y} \in \{0^+, \infty\}.$$

Suppose now that $t^R_{x,y} \in \mathbb{R}_{>0}$. We thus have $s^R_{x,y} = r^+$ for a $r \in \mathbb{R}_{>0}$, and

$$\mathcal{P}^{R^\vee}_{y,x} = \mathbb{Q}_{>0} \cap [r^{-1}, +\infty[.$$

We distinguish two cases: either $r \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$, and then $(y, R^\vee, x) \in \mathcal{A}^{R^\vee}$ and $t^{R^\vee}_{y,x} = r$; or $r \in \mathbb{Q}_{>0}$, and then $(y, R^\vee, x) \in \mathcal{B}^{R^\vee}$ and $t^{R^\vee}_{y,x} = r^+$.

Suppose finally that $t^R_{x,y} \in \mathbb{B} \setminus \mathbb{A}$. We thus have $s^R_{x,y} = q$ for a $q \in \mathbb{Q}_{>0}$, and

$$\mathcal{P}^{R^\vee}_{y,x} = \mathbb{Q}_{>0} \cap]q^{-1}, +\infty[.$$

Therefore, we have $(y, R^\vee, x) \in \mathcal{A}^{R^\vee}$ and $t^{R^\vee}_{y,x} = q$. □

By (3.4), for $R \in \mathcal{R}(A)$, we have

$$\mathcal{B}^{R^\vee} = \{(y, x) \in A \times A : s^R_{x,y} \in \{q^+ : q \in \mathbb{Q}_{>0}\}\}. \tag{3.5}$$

4. The function Φ_R

For $R \in \mathcal{R}(A)$, denote $\Phi_R : A \times A \to \mathbb{B}$ the function $(x, y) \mapsto t^R_{x,y}$. Following Section 3, for $R \in \mathcal{R}(A)$ and $(x, y) \in A \times A$, we have $(x, y) \in \mathcal{A}^R \Leftrightarrow \Phi_R(x, y) \in \mathbb{A}$; in particular, we have

$$R \in \mathcal{R}'(A) \Leftrightarrow \Phi_R(A \times A) \subset \mathbb{A}. \tag{4.1}$$

Proposition 4.2. *For $R \in \mathcal{R}(A)$, Φ_R is the unique function $\Phi : A \times A \to \mathbb{B}$ satisfying (for all $x, y \in A$ and all $m, n \in \mathbb{N}^*$):*

(1) $\Phi(mx, ny) = \frac{m}{n} \cdot \Phi(x, y)$;
(2) $x\,R\,y \Leftrightarrow \Phi(x, y) > 1$.

Conversely, any binary relation $R^\sharp$ on A such that there exists a function $\Phi^\sharp : A \times A \to \mathbb{B}$ satisfying (1) and (2), belongs to $\mathcal{R}(A)$.

Proof. The converse is straightforward, and for $R \in \mathcal{R}(A)$, the function Φ_R satisfies the conditions (1) and (2) of the proposition. Let $R \in \mathcal{R}(A)$, and let $\Phi, \Phi' : A \times A \to \mathbb{B}$ be two functions satisfying the conditions (1) and (2) of the proposition. Suppose that there exists a couple $(x, y) \in A \times A$ such that $\Phi'(x, y) \neq \Phi(x, y)$. By symmetry, we can suppose that $\Phi'(x, y) > \Phi(x, y)$. By Remark 1.1, there exist $p, q \in \mathbb{N}^*$ such that $pq^{-1}\Phi'(x, y) > 1 \geq pq^{-1}\Phi(x, y)$. Therefore $px\,R\,qy$ and $px\,(-R)\,qy$; contradiction. Hence Φ is unique. □

The functions $\Phi_{R_\emptyset}$ and Φ_{R_∞} are constant, given by

$$\Phi_{R_\emptyset} = 0^+,$$
$$\Phi_{R_\infty} = \infty.$$

And for $R \in \mathcal{R}(A)$ and $q \in \mathbb{Q}_{>0}$, we have

$$\Phi_{R^q} = q \cdot \Phi_R.$$

For $R \in \mathcal{R}(A)$, the function $\Phi_{R'} : A \times A \to \mathbb{A}$ is given by

$$\Phi_{R'}(x, y) = \begin{cases} \Phi_R(x, y) & \text{if } \Phi_R(x, y) \in \mathbb{A} \\ q & \text{if } \Phi_R(x, y) = q^+. \end{cases}$$

We thus have $\Phi_R \geq \Phi_{R'}$ with the equality if and only if $R = R'$. For $R \in \mathcal{R}'(A)$, let

$$\mathcal{R}(A)_R = \{S \in \mathcal{R}(A) : S' = R\}$$

be the fibre of the projection $\mathcal{R}(A) \to \mathcal{R}'(A)$ above R. We thus have

$$\mathcal{R}(A)_R = \{S \in \mathcal{R}(A) : \Phi_R(x, y) \in \{\Phi_S(x, y), \Phi_S(x, y)^+\}, \ \forall (x, y) \in A \times A\}.$$

Let $R \in \mathcal{R}(A)$. Following (3.3), for $x, y \in A$, we have $x\,R^\vee\,y$ if and only if $s^R_{y,x} > 1$. Define $\sigma^R_{y,x} \in \mathbb{B}$ by

$$\sigma^R_{y,x} = \begin{cases} s^R_{y,x} & \text{if } s^R_{x,y} \in \mathbb{Q}^\natural_{>0} \\ r & \text{if } s^R_{y,x} = r^+ \text{ with } r \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}. \end{cases}$$

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Then we have $\sigma^R_{mx,ny} = \frac{m}{n} \cdot \sigma^R_{x,y}$ and $x R^\vee y \Leftrightarrow \sigma^R_{y,x} > 1$. Therefore, the function $A \times A \to \mathbb{B},\ (x, y) \mapsto \sigma^R_{y,x}$ coincides with $\Phi_{R^\vee}$. And by (3.4), for $x,\ y \in A$, the triplet $(x, R^\vee, y)$ is super-Archimedean if and only $s^R_{y,x} \notin \{q^+ : q \in \mathbb{Q}_{>0}\}$; i.e. if and only if $t^R_{y,x} \notin \mathbb{Q}_{>0}$. We deduce that

$$R \in \mathcal{R}'(A)^\vee \Leftrightarrow \Phi_R(A \times A) \subset \mathbb{B} \setminus \mathbb{Q}_{>0}. \tag{4.3}$$

By (4.1) and (4.3), for $R \in \mathcal{R}(A)$, we have

$$R \in \mathcal{R}''(A) \Leftrightarrow \Phi_R(A \times A) \subset \mathbb{A} \setminus \mathbb{Q}_{>0}. \tag{4.4}$$

Let $\mathcal{R}^{\emptyset,\infty}(A)$ be the set of relations $R \in \mathcal{R}(A)$ such that $\Phi_R(A \times A) \subset \{0^+, \infty\}$. By (4.4), we have the inclusion

$$\mathcal{R}^{\emptyset,\infty}(A) \subset \mathcal{R}''(A).$$

Precisely, the involution $\mathcal{R}(A) \to \mathcal{R}(A),\ R \mapsto R^\vee$ induces by restriction a bijective map

$$\mathcal{R}^{\emptyset,\infty}(A) \to \mathcal{R}^{\emptyset,\infty}(A).$$

5. The functions $\Phi_>,\ \Phi_\geq : \mathbb{R}^\natural_{>0} \times \mathbb{R}^\natural_{>0} \to \mathbb{B}$

The relations $>$ and $\geq$ on the $\mathbb{R}_{>0}$-set $\mathbb{R}^\natural_{>0}$, are positive homothetic orders, and we have $\geq = (>)^\vee$. Let

$$\mathcal{Q} = \{(r, r') \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} : r'^{-1} r \in \mathbb{Q}_{>0}\}.$$

The subsets $\mathcal{B}^>$ and $\mathcal{B}^\geq$ of $\mathbb{R}^\natural_{>0} \times \mathbb{R}^\natural_{>0}$ are given by:

$$\mathcal{B}^> = \{(r^+, r') : (r, r') \in \mathcal{Q}\},$$
$$\mathcal{B}^\geq = \mathcal{B}^> \cup \mathcal{Q} \cup \{(r^+, r'^+) : (r, r') \in \mathcal{Q}\}.$$

And the subsets $\mathcal{B}^>_i$ and $\mathcal{B}^\geq_i$ $(i = 1,\ 2)$ of $\mathbb{R}^\natural_{>0}$ are given by:

$$\mathcal{B}^>_1 = \{r^+ : r \in \mathbb{R}_{>0}\},$$
$$\mathcal{B}^>_2 = \mathbb{R}_{>0},$$
$$\mathcal{B}^\geq_1 = \mathcal{B}^\geq_2 = \mathcal{B}^>_1 \cup \mathcal{B}^>_2.$$

Following (4.2), for $R \in \{>, \geq\}$, there exists a unique function $\Phi_R : \mathbb{R}^\natural_{>0} \times \mathbb{R}^\natural_{>0} \to \mathbb{B}$ satisfying (for all $r,\ r' \in \mathbb{R}^\natural_{>0}$ and all $m,\ n \in \mathbb{N}^*$):

(1) $\Phi_R(m \cdot r, n \cdot r') = \frac{m}{n} \cdot \Phi(r, r')$;
(2) $r > r' \Leftrightarrow \Phi_R(r, r') > 1$.

The function $\Phi_>$ is explicitly given by:

$$\Phi_>(r, \infty) = \Phi_>(0^+, r) = 0^+ \quad (r \in \mathbb{R}^\natural_{>0}),$$
$$\Phi_>(\infty, r) = \infty \quad (r \in \mathbb{R}^\natural_{>0} \setminus \{\infty\}),$$
$$\Phi_>(r, 0^+) = \infty \quad (r \in \mathbb{R}^\natural_{>0} \setminus \{0^+\}),$$
$$\Phi_>(r, r') = \Phi_>(r^+, r'^+) = \Phi_>(r, r'^+) = r'^{-1} r \quad (r,\ r' \in \mathbb{R}_{>0}),$$
$$\Phi_>(r^+, r') = r'^{-1} r \quad (r,\ r' \in \mathbb{R}_{>0},\ r'^{-1} r \notin \mathbb{Q}_{>0}),$$
$$\Phi_>(r^+, r') = (r'^{-1} r)^+ \quad (r,\ r' \in \mathbb{R}_{>0},\ r'^{-1} r \in \mathbb{Q}_{>0}).$$

And the function $\Phi_\geq$ is explicitly given by:

$$\Phi_\geq(\infty, r) = \Phi_\geq(r, 0^+) = \infty \quad (r \in \mathbb{R}^\natural_{>0}),$$
$$\Phi_\geq(r, \infty) = 0^+ \quad (r \in \mathbb{R}^\natural_{>0} \setminus \{\infty\}),$$
$$\Phi_\geq(0^+, r) = 0^+ \quad (r \in \mathbb{R}^\natural_{>0} \setminus \{0^+\}),$$
$$\Phi_\geq(r, r') = \Phi_\geq(r^+, r'^+) = \Phi_\geq(r^+, r') = r'^{-1} r \quad (r,\ r' \in \mathbb{R}_{>0},\ r'^{-1} r \notin \mathbb{Q}_{>0}),$$
$$\Phi_\geq(r, r') = \Phi_\geq(r^+, r'^+) = \Phi_\geq(r^+, r') = (r'^{-1} r)^+ \quad (r,\ r' \in \mathbb{R}_{>0},\ r'^{-1} r \in \mathbb{Q}_{>0}),$$
$$\Phi_\geq(r, r'^+) = r'^{-1} r \quad (r,\ r' \in \mathbb{R}_{>0}).$$

By the above formulas, for $R \in \{>, \geq\}$, $r,\ r' \in \mathbb{R}^\natural_{>0}$ and $a,\ b \in \mathbb{Q}_{>0}$, we have

$$\Phi_R(a \cdot r, b \cdot r') = b^{-1} a \cdot \Phi_R(r, r'). \tag{5.1}$$

Remark 5.2. Thanks to the above formulas, for $R \in \{>, \geq\}$, we can explicitly describe the relation $R' \in \mathcal{R}'(A)$. ⋆

Remark 5.3. The relations $>$ and $\geq$ induce by restriction two positive homothetic orders on the $\mathbb{Q}_{>0}$-set $\mathbb{B}$. And the functions $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$ associated to these two orders are of course the restrictions to $\mathbb{B} \times \mathbb{B}$ of the functions $\Phi_>$ and $\Phi_\geq$. ⋆

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6. Generalized homothetic biorders

Let R be a binary relation on A. We say that R is a *generalized homothetic biorder* if there exist two functions $u_1,\ u_2 : A \to \mathbb{R}^{\natural}_{>0}$ satisfying (for all $x,\ y \in A$ and all $m \in \mathbb{N}^*$):

(1) $u_i(mx) = m \cdot u_i(x)$ $(i = 1,\ 2)$;
(2) $x\,R\,y \Leftrightarrow u_1(x) > u_2(y)$.

Clearly, any generalized homothetic biorder on A is an element of $\mathcal{R}(A)$. And the relations $R_\emptyset$ and R_∞ are generalized homothetic biorders: for $R = R_\emptyset$, we can take for (u_1, u_2) the pair of constant functions $(0^+, \infty)$; and for $R = R_\infty$, we can take for (u_1, u_2) the pair of constant functions $(\infty, 0^+)$.

Let $\mathcal{R}_\bullet(A)$ be the subset of $\mathcal{R}(A)$ formed by generalized homothetic biorders. And let

$$\mathcal{R}'_\bullet(A) = \mathcal{R}_\bullet(A) \cap \mathcal{R}'(A),$$
$$\mathcal{R}^{\emptyset,\infty}_\bullet(A) = \mathcal{R}_\bullet(A) \cap \mathcal{R}^{\emptyset,\infty}(A).$$

We thus have the inclusions

$$\{R_\emptyset, R_\infty\} \subset \mathcal{R}^{\emptyset,\infty}_\bullet(A) \subset \mathcal{R}_\bullet(A) \cap \mathcal{R}''(A) \subset \mathcal{R}'_\bullet(A) \subset \mathcal{R}_\bullet(A).$$

Definition 6.1. Let $R \in \mathcal{R}_\bullet(A)$. We call *representation of* R a pair of functions (u_1, u_2) on A with values in $\mathbb{R}^{\natural}_{>0}$, satisfying the conditions (1) and (2) above. More generally, if $\mathbb{E}$ is a sub-$\mathbb{N}^*$-set of $\mathbb{R}^{\natural}_{>0}$, we call *representation of* R *in* $\mathbb{E}$ a pair of functions (u_1, u_2) on A with values in $\mathbb{E}$, satisfying the conditions (1) and (2) above.

Lemma 6.2. *Let* $R \in \mathcal{R}_\bullet(A)$*, and let* (u_1, u_2) *be a representation of* R*. For all* $x,\ y \in A$*, we have*

$$\Phi_R(x, y) = \Phi_>(u_1(x), u_2(y)).$$

Proof. Clear. □

For $R \in \mathcal{R}(A)$, we define

$$A^R_1 = \{x \in A : x\,R\,y,\ \exists y \in A\} = \{x \in A : \mathcal{P}^R_{x,y} \neq \emptyset,\ \exists y \in A\},$$
$$A^R_2 = \{y \in A : x\,R\,y,\ \exists x \in A\} = \{y \in A : \mathcal{P}^R_{x,y} \neq \emptyset,\ \exists x \in A\},$$

and

$$A^R_{1,2} = \{x \in A^R_1 : y\,R^\vee\,x,\ \exists y \in A\} = \{x \in A^R_1 : \mathcal{P}^R_{x,y} \neq \mathbb{Q}_{>0},\ \exists y \in A\},$$
$$A^R_{2,1} = \{y \in A^R_2 : y\,R^\vee\,x,\ \exists x \in A\} = \{y \in A^R_2 : \mathcal{P}^R_{x,y} \neq \mathbb{Q}_{>0},\ \exists x \in A\}.$$

We have the inclusion

$$\mathcal{B}^R \subset A^R_{1,2} \times A^R_{2,1}. \tag{6.3}$$

Notice that

$$A^R_1 = \emptyset \Leftrightarrow A^R_2 = \emptyset \Leftrightarrow R = R_\emptyset$$

and that

$$A^R_{1,2} = \emptyset \Leftrightarrow A^R_{2,1} = \emptyset \Leftrightarrow R \in \mathcal{R}^{\emptyset,\infty}(A).$$

Let $R \in \mathcal{R}_\bullet(A)$, and let (u_1, u_2) be a representation of R. Then we have

$$\mathcal{B}^R = \{(x, y) \in A \times A : (u_1(x), u_2(y)) \in \mathcal{B}^>\}.$$

We then deduce that

$$\mathcal{B}^R_1 = \{x \in A : u_1(x) = u_2(y)^+,\ \exists y \in A\},$$
$$\mathcal{B}^R_2 = \{y \in A : u_1(x) = u_2(y)^+,\ \exists x \in A\}.$$

If $R \neq R_\emptyset$, then $u_1 \neq 0^+$ and $u_2 \neq \infty$, and we have

$$A^R_1 = \{x \in A : u_1(x) \neq 0^+\},$$
$$A^R_2 = \{y \in A : u_2(y) \neq \infty\}.$$

And if $R \notin \mathcal{R}^{\emptyset,\infty}(A)$, then $u_i(A) \not\subset \{0^+, \infty\}$ $(i = 1,\ 2)$, and we have

$$A^R_{1,2} = \{x \in A : u_1(x) \notin \{0^+, \infty\}\},$$
$$A^R_{2,1} = \{y \in A : u_2(y) \notin \{0^+, \infty\}\}.$$

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Lemma 6.4. *Let $R \in \mathcal{R}_\bullet^{\emptyset,\infty}(A) \setminus \{R_\emptyset\}$. There exists a representation (u_1, u_2) of R in $\{0^+, \infty\}$, and this representation is unique.*

Proof. For $(x, y) \in A \times A$, we define

$$u_1(x) = \begin{cases} \infty & \text{if } x \in A_1^R \\ 0^+ & \text{otherwise} \end{cases}$$

and

$$u_2(y) = \begin{cases} 0^+ & \text{if } y \in A_2^R \\ \infty & \text{otherwise.} \end{cases}$$

Since $R \in \mathcal{R}_\bullet^{\emptyset,\infty}(A)$, we have $x\,R\,y \Leftrightarrow (x, y) \in A_1^R \times A_2^R$. Therefore, the pair (u_1, u_2) represents R. And since $R \neq R_\emptyset$, the sets A_1^R and A_2^R are nonempty, therefore (u_1, u_2) is the unique representation of R in $\{0^+, \infty\}$. □

Notice that in (6.4), without the condition $u_i(A) \subset \{0^+, \infty\}$ $(i = 1, 2)$, the uniqueness property is no longer true: for any two morphisms of $\mathbb{N}^*$-sets $u_1 : A \to \mathbb{A} \setminus \{0^+\}$ and $u_2 : A \to \mathbb{A} \setminus \{\infty\}$, the pairs $(u_1, 0^+)$ and (∞, u_2) represent R_∞. Note also that for $R = R_\emptyset$, the Lemma 6.4 is not true: the pairs $(0^+, 0^+)$, $(0^+, \infty)$ and (∞, ∞) represent $R_\emptyset$.

Lemma 6.5. *Let $R \in \mathcal{R}_\bullet(A) \setminus \mathcal{R}_\bullet^{\emptyset,\infty}(A)$. There exists a representation (u_1, u_2) of R such that for $i = 1, 2$, we have $u_i(\mathcal{A}_i^R) \subset \mathbb{A}$. In particular, we have $u_2(A) \subset \mathbb{A}$; and if $R \in \mathcal{R}'(A)$, then (u_1, u_2) is a representation of R in $\mathbb{A}$. Moreover, up to multiplication by an element of $\mathbb{R}_{>0}$, the pair (u_1, u_2) is unique: if (u'_1, u'_2) is another representation of R such that for $i = 1, 2$, we have $u'_i(\mathcal{A}_i^R) \subset \mathbb{A}$, then there exists a $\lambda \in \mathbb{R}_{>0}$ such that $(u'_1, u'_2) = (\lambda \cdot u_1, \lambda \cdot u_2)$.*

Proof. Let (v_1, v_2) be a representation of R. For $i = 1, 2$, let $u_i : A \to \mathbb{A}$ be the function defined by

$$u_i(x) = \begin{cases} v_i(x) & \text{if } v_i(x) \in \mathbb{A} \text{ or } x \in \mathcal{B}_i^R \\ r & \text{if } v_i(x) = r^+ \text{ and } x \in \mathcal{A}_i^R. \end{cases}$$

Since $\{(x, y) \in A \times A : u_1(x) = u_2(y)^+\} \subset \mathcal{B}_1^R \times \mathcal{B}_2^R$, the pair (u_1, u_2) is a representation of R. By construction, for $i = 1, 2$, we have $u_i(\mathcal{A}_i^R) \subset \mathbb{A}$. And since the set $\{y \in A : v_2(y) \in \mathbb{B} \setminus \mathbb{A}\}$ is contained in $\mathcal{A}_2^R$, we have $u_2(A) \subset \mathbb{A}$. Finally, if $R \in \mathcal{R}'(A)$, since $\mathcal{A}_1^R = A = \mathcal{A}_2^R$, (u_1, u_2) is a representation of R in $\mathbb{A}$.

Let (u'_1, u'_2) be another representation of R such that for $i = 1, 2$, we have $u'_i(\mathcal{A}_i^R) \subset \mathbb{A}$. By (6.2), for $x, y \in A$, we have

$$\Phi_>(u_1(x), u_2(y)) = \Phi_R(x, y) = \Phi_>(u'_1(x), u'_2(y)).$$

Since $R \notin \mathcal{R}_\bullet^{\emptyset,\infty}(A)$, we have $u_i(A) \not\subset \{0^+, \infty\}$ $(i = 1, 2)$, and hence:

- on $A \setminus A_1^R$, we have $u_1 = u'_1 = 0^+$;
- on $A_1^R \setminus A_{1,2}^R$, we have $u_1 = u'_1 = \infty$;
- on $A \setminus A_2^R$, we have $u_2 = u'_2 = \infty$;
- on $A_2^R \setminus A_{2,1}^R$, we have $u_2 = u'_2 = 0^+$.

On the other hand, for $(x, y) \in (\mathcal{A}_1^R \times \mathcal{A}_2^R) \cap (A_{1,2}^R \times A_{2,1}^R)$, we have $u_1(x), u_2(y), u'_1(x), u'_2(y) \in \mathbb{R}_{>0}$ and

$$u'_2(y)^{-1}u'_1(x) = \Phi_R(x, y) = u_2(y)^{-1}u_1(x).$$

Therefore, if $\mathcal{A}_1^R \times \mathcal{A}_2^R \neq \emptyset$, then there exists a constant $\lambda \in \mathbb{R}_{>0}$ such that for all $(x, y) \in \mathcal{A}_1^R \times \mathcal{A}_2^R$, we have

$$(u'_1(x), u'_2(y)) = (\lambda \cdot u_1(x), \lambda \cdot u_2(y)).$$

In particular, if $R \in \mathcal{R}'(A)$, the lemma is proved.

Suppose now that $R \notin \mathcal{R}'(A)$. Then the set $\mathcal{B}^R = \Phi_R^{-1}(\mathbb{B} \setminus \mathbb{A})$ is nonempty. And for $(x, y) \in \mathcal{B}^R$, we have $u_1(x) \in \mathbb{R}_{>0}^\natural \setminus \mathbb{A}$, $u_2(y) \in \mathbb{R}_{>0}$, and $u_2(y)^{-1} \cdot u_1(x) = q^+$ for a $q \in \mathbb{Q}_{>0}$; in particular, we have $\Phi_R(x, y) = u_2(y)^{-1} \cdot u_1(x)$. For $(x, y) \in \mathcal{B}^R$, we thus have

$$u_2(y)^{-1} \cdot u_1(x) = u'_2(y)^{-1} \cdot u'_1(x) \in \{q^+ : q \in \mathbb{Q}_{>0}\}.$$

Therefore, there exists a constant $\mu \in \mathbb{R}_{>0}$ such that for all $(x, y) \in \mathcal{B}^R$, we have

$$(u'_1(x), u'_2(y)) = (\mu \cdot u_1(x), \mu \cdot u_2(y)).$$

In particular, if $\Phi_R^{-1}(\mathbb{R}_{>0}) = \emptyset$, the lemma is proved.

Take a couple $(x, y) \in \Phi_R^{-1}(\mathbb{R}_{>0}) \subset \mathcal{A}^R$, and let us show that $u'_1(x) = \mu \cdot u_1(x)$ and $u'_2(y) = \mu \cdot u_2(y)$. If $x \in \mathcal{B}_1^R$, then there exists a $b \in A$ such that $(x, b) \in \mathcal{B}^R$; and we have $u'_1(x) = \mu \cdot u_1(x)$. Suppose that $x \in \mathcal{A}_1^R$. Then $u_1(x), u'_1(x) \in \mathbb{R}_{>0}$. If $u'_1(x) > \mu u_1(x)$, then there exist $m, n \in \mathbb{N}^*$ such that

$$u'_1(mx) > u'_2(nb) = \mu u_2(nb) > \mu u_1(mx);$$

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contradiction. Also, if $u_1'(x) < \mu u_1(x)$, then there exist $m,\ n \in \mathbb{N}^*$ such that

$$u_1'(mx) < u_2'(nb) = \mu u_2(nb) < \mu u_1(mx);$$

contradiction. Hence $u_1'(x) = \mu u_1(x)$. The equality $u_2'(y) = \mu \cdot u_2(y)$ is obtained similarly. This ends the proof of the lemma. □

Notice that in (6.5), it follows from the above proof that without the condition $u_i(\mathcal{A}_i^R) \subset \mathbb{A}$ $(i = 1,\ 2)$, the uniqueness property is no longer true.

Lemma 6.6. *Let $R \in \mathcal{R}_\bullet(A)$, and let (u_1, u_2) be a representation of R. We have:*

$$R \in \mathcal{R}_\bullet^{\emptyset,\infty}(A) \Leftrightarrow u_1(A) \subset \{0^+, \infty\} \text{ or } u_2(A) \subset \{0^+, \infty\},$$
$$R \in \mathcal{R}_\bullet'(A) \Leftrightarrow u_1(A) \cap \{r^+ : r \in u_2(A) \cap \mathbb{R}_{>0}\} = \emptyset.$$

Proof. For $(x, y) \in A \times A$, we have $\Phi_R(x, y) \in \{0^+, \infty\}$ if and only if $u_1(x) \in \{0^+, \infty\}$ or $u_2(y) \in \{0^+, \infty\}$; and we have $(x, y) \in \mathcal{B}^R$ if and only there exists a $q \in \mathbb{Q}_{>0}$ such that $u_1(x) = q \cdot u_2(y)^+$. This ends the proof of the lemma. □

7. Generalized homothetic intervals (resp. weak) orders

A generalized homothetic biorder on A is called a:

- *generalized homothetic interval order* if for any (i.e. for one) representation (u_1, u_2) of R, we have $u_1 \le u_2$;
- *generalized homothetic weak order* if for any (i.e. for one) representation (u_1, u_2) of R, we have $u_1 = u_2$; in which case we say that u_1 is a representation of R.

Let us recall that a relation binary R on A is said to be:

- *reflexive* if for all $x \in A$, we have $x\,R\,x$;
- *symmetric* if for all $x,\ y \in A$, we have $x\,R\,y \Leftrightarrow y\,R\,y$;
- *transitive* if for all $x,\ y,\ z \in A$, we have $x\,R\,y\,R\,z \Rightarrow x\,R\,z$.

For all binary relations R and R' on A, we note $R \cap R'$ the binary relation on A defined by

$$x\,(R \cap R')\,y \Leftrightarrow x\,R\,y \text{ and } x\,R'\,y.$$

Note that for any binary relation R on A, the *indifference relation* $S = R^\vee \cap (-R)$ is symmetric. Indeed, for $x,\ y \in A$, we have

$$x\,S\,y \Leftrightarrow x\,(-R)\,y \text{ and } y\,(-R)\,x.$$

Lemma 7.1. *Let $R \in \mathcal{R}_\bullet(A)$, and let $S = R^\vee \cap (-R)$. Then R is:*

- *a generalized homothetic interval order if and only if S is reflexive;*
- *a generalized homothetic weak order if and only if S is reflexive and transitive.*

Proof. Let (u_1, u_2) be a representation of R. For $x,\ y \in A$, we have

$$x\,S\,y \Leftrightarrow \begin{cases} u_2(x) \ge u_1(y) \\ u_2(y) \ge u_1(x). \end{cases}$$

Therefore, S is reflexive if and only if $u_1 \le u_2$; i.e. if and only if R is a generalized homothetic interval order.

Suppose that R is a generalized homothetic weak order. Then $u_1 = u_2$, and for all $x,\ y \in A$, we have $x\,S\,y \Leftrightarrow u_1(x) = u_1(y)$. Therefore S is transitive.

Conversely, suppose that S is reflexive and transitive. Suppose that there exists a $x \in A$ such that $u_1(x) \ne u_2(x)$. Since $x\,S\,x$, we have $u_1(x) < u_2(x)$. Let $q,\ q' \in \mathbb{Q}_{>0}$ such that $q < 1 < q'$ and

$$qu_1(x) < u_1(x) < qu_2(x) < q'u_1(x) < u_2(x) < q'u_2(x).$$

Write $q = \frac{m}{n}$ and $q' = \frac{m'}{n'}$ with $m,\ n,\ m',\ n' \in \mathbb{N}^*$, and let $y = nn'x$, $z = mn'x$ and $t = m'nx$. Then we have

$$u_1(z) < u_1(y) < u_2(z) < u_1(t) < u_2(y) < u_2(t).$$

Hence $z\,S\,y\,S\,t$ and $z\,(-S)\,t$; contradiction. Therefore, $u_1 = u_2$. □

Remark 7.2. The relation $>$ on $\mathbb{R}_{>0}^\natural$ is a generalized homothetic weak order: it is represented by the identity morphism $\mathbb{R}_{>0}^\natural \to \mathbb{R}_{>0}^\natural$. ⋆

Remark 7.3. The empty relation on A is a generalized homothetic weak order: the constant functions $u = 0^+$ and $u = \infty$ represent $R_\emptyset$. On the contrary, the trivial relation on A *is neither* a generalized homothetic weak order nor a generalized homothetic interval order: for all representations (u_1, u_2) of R_∞, we have $u_1 > u_2$. ⋆

8. An example: The relation $\succ$ on $T(A)$

Let $T(A) = A \times \mathcal{R}(A) \times A$. We endow $T(A)$ with the structure of a $\mathbb{Q}_{>0}$-set (hence a fortiori of a $\mathbb{N}^*$-set) defined by $q \cdot (x, R, y) = (x, R^q, y)$, and we note $\succ$ the binary relation on $T(A)$ defined by: $(x_1, R_1, y_1) \succ (x_2, R_2, y_2)$ if and only there exist $m,\ n \in \mathbb{N}^*$ such that $mx_1\, R_1\, ny_1$ and $mx_2\, (-R_2)\, ny_2$.

Lemma 8.1. *The relation $\succ$ on $T(A)$ is a generalized homothetic weak order. Moreover, the function $u_\succ : T(A) \to \mathbb{B}$ given by $u_\succ(x, R, y) = \Phi_R(x, y)$ is a representation of $\succ$ in $\mathbb{B}$, and a morphism of $\mathbb{Q}_{>0}$-sets.*

Proof. The function Φ is clearly a morphism of $\mathbb{Q}_{>0}$-sets. Let $(x_1, R_1, y_1),\ (x_2, R_2, y_2) \in T(A)$. We have $(x_1, R_1, y_1) \succ (x_2, R_2, y_2)$ if and only if there exist $m,\ n \in \mathbb{N}^*$ such that $\frac{m}{n} \cdot \Phi_{R_1}(x, y) > 1 \geq \frac{m}{n} \cdot \Phi_{R_2}(x, y)$; i.e. if and only there exists a $q \in \mathbb{Q}_{>0}$ such that $\Phi_{R_1}(x, y) > q \geq \Phi_{R_2}(x, y)$. By (1.1), we obtain that

$$(x_1, R_1, y_1) \succ (x_2, R_2, y_2) \Leftrightarrow \Phi_{R_1}(x, y) > \Phi_{R_2}(x, y).$$

Hence the lemma. □

Let $\succsim$ the binary relation on $T(A)$ defined by $\succsim = \succ^\vee$. It is given by

$$(x_1, R_1, y_1) \succsim (x_2, R_2, y_2) \Leftrightarrow u_\succ(x_1, R_1, y_1) \geq u_\succ(x_2, R_2, y_2).$$

Let $\sim$ be the indifference relation associated with $\succ$, defined by

$$(x_1, R_1, y_1) \sim (x_2, R_2, y_2) \Leftrightarrow u_\succ(x_1, R_1, y_1) = u_\succ(x_2, R_2, y_2).$$

This is an equivalence relation on $T(A)$. The quotient set

$$\overline{T}(A) = T(A)/\sim$$

inherits the $\mathbb{Q}_{>0}$-set structure of $T(A)$, and $u_\succ : T(A) \to \mathbb{B}$ is factorized through an *injective* morphism of $\mathbb{Q}_{>0}$-sets

$$\overline{u}_\succ : \overline{T}(A) \hookrightarrow \mathbb{B}.$$

The study of the properties of this morphism will be the subject of a further work.

9. Characterization of generalized homothetic biorders

Consider the six following properties (for all $x,\ y,\ z,\ t \in A$):

($_1$S) if $(x, t) \in \mathcal{A}^R$ and $x\, R\, y\, R^\vee\, z\, R\, t$, then we have $x\, R\, t$;
($_2$S) if $(x, t) \in \mathcal{A}^R$, $(y, z) \in A^R_{2,1} \times A^R_{1,2}$ and $x\, R\, t$, then there exist $m, n, p \in \mathbb{N}^*$ such that we have $mx\, R\, ny\, R^\vee\, pz\, R\, mt$;
($_3$S) if $(x, t) \in \mathcal{B}^R$ and $t\, R^\vee\, y\, R\, z\, R^\vee\, x$, then we have $t\, R^\vee\, x$;
($_4$S) if $(x, t) \in \mathcal{B}^R$, $(y, z) \in A^R_{1,2} \times A^R_{2,1}$ and $t\, R^\vee\, x$, then there exist $m, n, p \in \mathbb{N}^*$ such that we have $mt\, R^\vee\, ny\, R\, pz\, R^\vee\, mx$;
($_5$S) The fibre $\Phi_R^{-1}(0^+)$ is empty or union of sets of the form $\{x\} \times A$ or $A \times \{y\}$, and the fibre $\Phi_R^{-1}(\infty)$ is empty or union of sets of the form $\{x\} \times A^R_2$ or $A^R_1 \times \{y\}$;
($_6$S) If $(x, y) \in \mathcal{B}^R_1 \times \mathcal{B}^R_2$, then we have $(y, x) \in \mathcal{A}^{R^\vee}$.

Remark 9.1. In the condition ($_6$S), we can replace the set $\mathcal{B}^R_1 \times \mathcal{B}^R_2$ by the set $(\mathcal{B}^R_1 \times \mathcal{B}^R_2) \cap \mathcal{A}^R$. Indeed, by (3.4), we know that for $(x, y) \in \mathcal{B}^R$, we have $(y, x) \in \mathcal{A}^{R^\vee}$. For $(x, y) \in \mathcal{B}^R_1 \times \mathcal{B}^R_2$, the triplet (x, R, y) is *potentially non super-Archimedean* in the sense that there exist $x',\ y' \in A$ such that the triplets (x, R, y') and (x', R, y) are non super-Archimedean. And the condition ($_6$S) means that the triplets potentially non super-Archimedean (x, R, y) behave as "true" non super-Archimedean triplets. □

Lemma 9.2. *Let $R \in \mathcal{R}(A)$. If R satisfies ($_5$S), then we have*

$$\Phi_R^{-1}(\mathbb{B} \setminus \{0^+, \infty\}) = A^R_{1,2} \times A^R_{2,1}.$$

Proof. For $x,\ y \in A$, we have

$$x \in A \setminus A^R_1 \Leftrightarrow \Phi_R(x, A) = 0^+,$$
$$y \in A \setminus A^R_2 \Leftrightarrow \Phi_R(A, y) = 0^+,$$

and

$$x \in A^R_1 \setminus A^R_{1,2} \Leftrightarrow \Phi_R(x, A^R_2) = \infty,$$
$$y \in A^R_2 \setminus A^R_{2,1} \Leftrightarrow \Phi_R(A^R_1, y) = \infty.$$

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If R satisfies $({}_5\mathrm{S})$, we deduce that

$$\Phi_R(x,y)=0^+ \Leftrightarrow x\in A\setminus A_1^R \text{ or } y\in A\setminus A_2^R,$$
$$\Phi_R(x,y)=\infty \Leftrightarrow x\in A_1^R\setminus A_{1,2}^R \text{ or } y\in A_2^R\setminus A_{2,1}^R.$$

Hence the lemma. □

Proposition 9.3. *For $R\in\mathcal{R}(A)$, we have $R\in\mathcal{R}_\bullet(A)$ if and only if R satisfies the properties $({}_i\mathrm{S})$ for $i=1,\dots,6$.*

Proof. Let $R\in\mathcal{R}_\bullet(A)$, and let (u_1,u_2) be a representation of R. Then the relation $R^\vee$ on A is given by

$$x\,R^\vee\,y \Leftrightarrow u_2(x)\geq u_1(y).$$

By looking at description of the sets $\mathcal{B}^R$, $A_{1,2}^R$ and $A_{2,1}^R$ given in the Section 6, it is easy to verify that the properties $({}_1\mathrm{S})$, $({}_2\mathrm{S})$, $({}_3\mathrm{S})$ and $({}_4\mathrm{S})$, are true for R. For $x,\ y\in A$, we have $\Phi_R(x,y)=0^+$ if and only if $u_1(x)=0^+$ or $u_2(x)=\infty$; and we have $\Phi_R(x,y)=\infty$ if and only if one of the two following conditions is satisfied:

- $u_2(x)=\infty$ and $u_1(y)\neq\infty$;
- $u_1(x)\neq 0^+$ and $u_1(y)=0^+$.

Therefore R satisfies $({}_5\mathrm{S})$. As for the property $({}_6\mathrm{S})$, let $(x,y)\in\mathcal{B}_1^R\times\mathcal{B}_2^R$. By Remark 9.1, we can suppose that $(x,y)\in\mathcal{A}^R$. By Section 6, we have $u_1(x)=r^+$ and $u_2(y)=r'$ for some $r,\ r'\in\mathbb{R}_{>0}$. And by (3.4), we have $(y,x)\in\mathcal{A}^{R^\vee}$ if and only if $t_{x,y}^R\in\mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}$. But by (6.2) and the Section 5, we have

$$t_{x,y}^R=\begin{cases} r'^{-1}r & \text{if } r'^{-1}r\in\mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}\\ (r'^{-1}r)^+ & \text{if } r'^{-1}r\in\mathbb{Q}_{>0}.\end{cases}$$

But the case $r'^{-1}r\in\mathbb{Q}_{>0}$ is not possible, because $(x,y)\in\mathcal{A}^R$. Hence R satisfies $({}_6\mathrm{S})$.

Conversely, let $R\in\mathcal{R}(A)$ be a relation satisfying the properties $({}_i\mathrm{S})$ for $i=1,\dots,6$. We can suppose that $R\neq R_\emptyset$. Then $A_1^R\neq\emptyset$ and $A_2^R\neq\emptyset$. By $({}_5\mathrm{S})$, for $(x,y)\in\Phi_R^{-1}(\{0^+,\infty\})$, we can let

$$u_1(x)=\begin{cases} 0^+ & \text{if } \Phi_R(x,A)=0^+\\ \infty & \text{if } \Phi_R(x,A_2^R)=\infty\end{cases}$$

and

$$u_2(y)=\begin{cases} 0^+ & \text{if } \Phi_R(A_1^R,y)=\infty\\ \infty & \text{if } \Phi_R(A,y)=0^+.\end{cases}$$

The function $u_1\times u_2$ on $\Phi_R^{-1}(\{0^+,\infty\})$ is well-defined, and for $(x,y)\in\Phi_R^{-1}(\{0^+,\infty\})$, we have

$$x\,R\,y\Leftrightarrow u_1(x)=\infty>0^+=u_2(y).$$

In particular if $R\in\mathcal{R}^{\emptyset,\infty}(A)$, then $\Phi_R^{-1}(\{0^+,\infty\})=A\times A$, the functions $u_1,\ u_2:A\to\{0^+,\infty\}$ are morphisms of $\mathbb{N}^*$-sets, and the relation R is a generalized homothetic biorder.

We now suppose that $R\notin\mathcal{R}^{\emptyset,\infty}(A)$. Then $A_{1,2}^R\neq\emptyset$ and $A_{2,1}^R\neq\emptyset$. And by (9.2), we have

$$\Phi_R^{-1}(\mathbb{B}\setminus\{0^+,\infty\})=A_{1,2}^R\times A_{2,1}^R.$$

We hence need to extend the function $u_1\times u_2$ on $A_{1,2}^R\times A_{2,1}^R$. Let a couple $(a,b)\in A_{1,2}^R\times A_{2,1}^R$. For $(x,y)\in\mathcal{A}^R$, By $({}_1\mathrm{S})$ and $({}_2\mathrm{S})$, we have the equality (cf. [13] Lemma 3.4)

$$\mathcal{P}_{x,y}^R=\mathcal{P}_{x,b}^R\mathcal{P}_{b,a}^{R^\vee}\mathcal{P}_{a,y}^R. \qquad (*)$$

Also, for $(x,y)\in\mathcal{B}^R$, by $({}_3\mathrm{S})$ and $({}_4\mathrm{S})$, as in the proof of Lemma 3.4 of [13], we obtain the equality

$$\mathcal{P}_{y,x}^{R^\vee}=\mathcal{P}_{y,a}^{R^\vee}\mathcal{P}_{a,b}^R\mathcal{P}_{b,x}^{R^\vee}. \qquad (**)$$

Suppose first that $R\in\mathcal{R}'(A)\setminus\mathcal{R}^{\emptyset,\infty}(A)$. Then $(a,b)\in\mathcal{A}^R$, therefore $s_{a,b}^R=r^+$ for a $r\in\mathbb{R}_{>0}$; and by (3.4), we have

$$t_{b,a}^{R^\vee}=\begin{cases} r & \text{if } r\in\mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}\\ r^+ & \text{if } r\in\mathbb{Q}_{>0}.\end{cases}$$

In particular, (b,a) is an element of $A_{1,2}^{R^\vee}\times A_{2,1}^{R^\vee}$. Let $(x,y)\in A_{1,2}^R\times A_{2,1}^R$. Since (x,b) and (x,a) are elements of $A_{1,2}^R\times A_{2,1}^R$, $t_{x,b}^R$ and $t_{a,y}^R$ are elements of $\mathbb{R}_{>0}$, and by $(*)$, we have the equality

$$t_{x,y}^R=t_{x,b}^R\,r\,t_{a,y}^R\in\mathbb{R}_{>0}.$$

Let

$$u_1(x) = t^R_{x,b}$$

and

$$u_2(y) = r^{-1}(t^R_{a,y})^{-1}.$$

Then we have

$$x\,R\,y \Leftrightarrow t^R_{x,y} > 1 \Leftrightarrow u_1(x) > u_2(y).$$

The functions $u_1,\ u_2 : A \to \mathbb{A}$ thereby defined are morphisms of $\mathbb{N}^*$-modules, and the relation R is a generalized homothetic biorder.

Suppose now that $R \in \mathcal{R}(A) \smallsetminus \mathcal{R}'(A)$. Then $\mathcal{B}^R\ (\subset A^R_{1,2} \times A^R_{2,1}) \neq \emptyset$, and we can suppose that the pair (a, b) has been chosen such that:

- $(a, b) \in \mathcal{A}^R$ if the inclusion $\mathcal{B}^R \subset A^R_{1,2} \times A^R_{2,1}$ is strict;
- $t^R_{a,b} \in \mathbb{Q}_{>0}$ if the set $\{(a', b') \in \mathcal{A}^R : t^R_{a',b'} \in \mathbb{Q}_{>0}\}$ is nonempty.

By (3.4), three cases may appear:

- **case 1**: $t^{R^\vee}_{b,a} = s^R_{a,b} = q \in \mathbb{Q}_{>0}$ if $(a, b) \in \mathcal{B}^R$;
- **case 2**: $t^{R^\vee}_{b,a} = (t^R_{a,b})^{-1} = r \in \mathbb{R}_{>0} \smallsetminus \mathbb{Q}_{>0}$ if $(a, b) \in \mathcal{A}^R$ and $t^R_{a,b} \notin \mathbb{Q}_{>0}$;
- **case 3**: $t^{R^\vee}_{b,a} = s^R_{a,b} = q^+ \in \mathbb{B} \smallsetminus \mathbb{A}$ if $(a, b) \in \mathcal{A}^R$ and $t^R_{a,b} \in \mathbb{Q}_{>0}$.

Denote $\mathbb{R}^\natural_{>0} \to \mathbb{A},\ r \to \tilde{r}$ the projection defined by

$$\tilde{r} = \begin{cases} r & \text{if } r \in \mathbb{A} \\ s & \text{if } r = s^+ \text{ for a } s \in \mathbb{R}_{>0}. \end{cases}$$

By (3.4), for $(x, y) \in A^R_{1,2} \times A^R_{2,1}$, we have $\tilde{t}^{R^\vee}_{y,x} = \tilde{s}^R_{x,y}$. Take $(x, y) \in A^R_{1,2} \times A^R_{2,1}$. let $\alpha = t^R_{x,b}$, $\beta = s^R_{a,y}$. By $(*)$, if $(x, y) \in \mathcal{A}^R$ (which excludes case 1), we have

$$t^R_{x,y} = \begin{cases} r\tilde{\alpha}\tilde{\beta}^{-1} & \text{in case 2} \\ q\tilde{\alpha}\tilde{\beta}^{-1} & \text{in case 3.} \end{cases}$$

And by $(**)$, if $(x, y) \in \mathcal{B}^R$, we have

$$t^R_{x,y} = \begin{cases} q^+ \cdot \tilde{\alpha}\tilde{\beta}^{-1} & \text{in case 1} \\ r^+ \cdot \tilde{\alpha}\tilde{\beta}^{-1} & \text{in case 2} \\ q^+ \cdot \tilde{\alpha}\tilde{\beta}^{-1} & \text{in case 3.} \end{cases}$$

Let us show that

$$t^R_{x,y} > 1 \Leftrightarrow \begin{cases} q^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 1} \\ r^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 2} \\ q\tilde{\alpha} > \tilde{\beta} & \text{in case 3 if } x \in \mathcal{A}^R_1 \text{ or } y \in \mathcal{A}^R_2 \\ q^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 3 if } (x, y) \in \mathcal{B}^R_1 \times \mathcal{B}^R_2. \end{cases}$$

Case 1 is obvious.

In case 2, if $(x, y) \in \mathcal{A}^R$, we have $t^R_{x,y} > 1 \Leftrightarrow r\tilde{\alpha} > \tilde{\beta}$; and if $(x, y) \in \mathcal{A}^R$ and $r\tilde{\alpha} = \tilde{\beta}$, then $t^R_{x,y} = 1$, which is impossible (since we are in case 2).

Suppose that we are in case 3. If $(x, y) \in \mathcal{B}^R\ (\subset \mathcal{B}^R_1 \times \mathcal{B}^R_2)$, we have $t^R_{x,y} > 1 \Leftrightarrow q^+ \cdot \tilde{\alpha} > \tilde{\beta}$. Suppose then that $(x, y) \in \mathcal{A}^R$. If $x \in \mathcal{A}^R_1$ or $y \in \mathcal{A}^R_2$, then we have $t^R_{x,y} > 1 \Leftrightarrow q\tilde{\alpha} > \tilde{\beta}$. There remains the case $(x, y) \in \mathcal{B}^R_1 \times \mathcal{B}^R_2$. This is when we use property (${}_6$S): we have $t^R_{x,y} > 1 \Leftrightarrow q\tilde{\alpha} > \tilde{\beta}$; and if $q\tilde{\alpha} = \tilde{\beta}$, then $t^R_{x,y} = q\tilde{\alpha}\tilde{\beta}^{-1} = 1 \in \mathbb{Q}_{>0}$, therefore (by (3.4)) $(y, x) \notin \mathcal{A}^{R^\vee}$, which contradicts property (${}_6$S). We thus have $t^R_{x,y} > 1 \Leftrightarrow q^+ \cdot \tilde{\alpha} > \tilde{\beta}$.

Let

$$u_1(x) = \begin{cases} q^+ \cdot \tilde{t}^R_{x,b} & \text{in case 1} \\ r^+ \cdot \tilde{t}^R_{x,b} & \text{in case 2} \\ q\tilde{t}^R_{x,b} & \text{in case 3 if } x \in \mathcal{A}^R_1 \\ q^+ \cdot \tilde{t}^R_{x,b} & \text{in case 3 if } x \in \mathcal{B}^R_1 \end{cases}$$

and

$$u_2(y) = \tilde{s}^R_{a,y}.$$

Then we have

$$x\,R\,y \Leftrightarrow t^R_{x,y} \Leftrightarrow u_1(x) > u_2(y).$$

The functions $u_1,\ u_2 : A \to \mathbb{A}$ thereby defined are morphisms of $\mathbb{N}^*$-modules, and the relation R is a generalized homothetic biorder. This ends the proof of the proposition. □

Remark 9.4. For $R \in \mathcal{R}'(A)$, the properties $({}_3\mathrm{S})$, $({}_4\mathrm{S})$ and $({}_6\mathrm{S})$ are empty. Therefore, properties $({}_1\mathrm{S})$, $({}_2\mathrm{S})$ and $({}_5\mathrm{S})$ characterize the relations $R \in \mathcal{R}'_\bullet(A)$. ⋆

Remark 9.5. In general, the inclusion $\mathcal{R}_\bullet(A) \subset \mathcal{R}(A)$ is strict. For instance, take for A the union $\mathbb{N}^*x \coprod \mathbb{N}^*y$ of two copies of $\mathbb{N}^*$, endowed with the natural structure of $\mathbb{N}^*$-set, and let R stands for the binary relation on A defined by (for $m,\ n \in \mathbb{N}^*$):

- $mx\,R\,nx \Leftrightarrow m > n$;
- $mx\,R\,ny$ for all $m,\ n$;
- $my\,R\,ny \Leftrightarrow m > n$;
- $my\,(-R)\,nx$ for all $m,\ n$.

The relation R is h-independent and h-positive, but it is not a generalized homothetic biorder. ⋆

Remark 9.6. The positive homothetic order $\geq$ on $\mathbb{R}^\natural_{>0}$ is not a generalized homothetic biorder. Indeed, the property $({}_6\mathrm{S})$ is not satisfied: for $r,\ r' \in \mathbb{R}_{>0}$ such that $r'^{-1}r \in \mathbb{Q}_{>0}$, we have $(r', r^+) \in (\mathcal{B}^R_1 \times \mathcal{B}^R_2) \setminus \mathcal{B}^R$ and $(r^+, r') \in \mathcal{B}^>$. ⋆

10. "Operations" on generalized homothetic biorders

Let us consider the projection $\mathbb{R}^\natural_{>0} \to \mathbb{A},\ r \to \tilde{r}$ defined in the proof of (9.3). And for any function $u : A \to \mathbb{R}^\natural_{>0}$, denote $\tilde{u} : A \to \mathbb{A}$ the function defined by $\tilde{u}(x) = \widetilde{u(x)}$.

Let $R \in \mathcal{R}_\bullet(A)$, and let (u_1, u_2) be a representation of R. For $q \in \mathbb{Q}_{>0}$, the positive homothetic order R^q is a generalized homothetic biorder represented by the pair of functions $(q{\cdot}u_1, u_2)$. Similarly, the positive homothetic order R' is a generalized homothetic biorder represented by the pair of functions $(\tilde{u}_1, \tilde{u}_2)$. As for the order $R^\vee$, for $x,\ y \in A$, we have

$$x\,R^\vee\,y \Leftrightarrow u_2(x) \geq u_1(y).$$

Lemma 10.1. *Let $R \in \mathcal{R}_\bullet(A)$, and let (u_1, u_2) be a representation of R. For all $x,\ y \in A$, we have:*

$$\Phi_{R^\vee}(x, y) = \Phi_{\geq}(u_2(x), u_1(y)).$$

Proof. Clear. □

Note $\mathcal{R}_\bullet(A)^\vee$ the subset of $\mathcal{R}(A)$ formed by orders R such that there exist two functions $v_1,\ v_2 : A \to \mathbb{R}^\natural_{>0}$ satisfying (for all $x,\ y \in A$ and all $m \in \mathbb{N}^*$):

(1) $v_i(mx) = mv_i(x)$ $(i = 1,\ 2)$;
(2) $x\,R\,y \Leftrightarrow v_1(x) \geq v_2(y)$.

Let $R \in \mathcal{R}_\bullet(A)^\vee$, and let (u_1, u_2) be a representation of $R^\vee$. Then we have

$$\mathcal{B}^R = \{(x, y) \in A \times A : (u_2(x), u_1(y)) \in \mathcal{B}^{\geq}\}.$$

The involution $\mathcal{R}(A) \to \mathcal{R}(A),\ R \mapsto R^\vee$ induces by restriction two bijective maps

$$\mathcal{R}_\bullet(A) \to \mathcal{R}_\bullet(A)^\vee,$$
$$\mathcal{R}_\bullet(A)^\vee \to \mathcal{R}_\bullet(A),$$

that are inverse one another.

Remark 10.2. Directly or through the bijection $\mathcal{R}_\bullet(A) \to \mathcal{R}_\bullet(A)^\vee$, one can characterize the relations $R \in \mathcal{R}_\bullet(A)^\vee$ as in Section 9. But we will not do it here. It is also possible to characterize the relations $R \in \mathcal{R}_\bullet(A)$ such that $R^\vee \in \mathcal{R}_\bullet(A)$, as in Section 9 or in terms of a representation (u_1, u_2) of R (the following result is given without proof):

(1) *Let $R \in \mathcal{R}^{\emptyset,\infty}_\bullet(A) \setminus \{R_\emptyset\}$, and let (u_1, u_2) be the representation of R in $\{0^+, \infty\}$. Then $R^\vee \in \mathcal{R}_\bullet(A)$ if and only if $u_1 = \infty$ or $u_2 = 0^+$.*
(2) *Let $R \in \mathcal{R}_\bullet(A) \setminus \mathcal{R}^{\emptyset,\infty}_\bullet(A)$, and let (u_1, u_2) be a representation of R. Put $X = u_1(A) \cap u_2(A)$. Then $R^\vee \in \mathcal{R}_\bullet(A)$ if and only if $X \cap \{0^+, \infty\} = \emptyset$ and $X \cap \{r^+ : r \in X \cap \mathbb{R}_{>0}\} = \emptyset$. Moreover, if $R \in \mathcal{R}'_\bullet(A)$ and (u_1, u_2) is a representation in $\mathbb{A}$, then $R^\vee \in \mathcal{R}_\bullet(A)$ if and only if $X \cap \{0^+, \infty\} = \emptyset$, and $R^\vee \in \mathcal{R}'_\bullet(A)$ if and only if $X = \emptyset$.* ⋆

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11. The relation R_1 for $R \in \mathcal{R}_\bullet(A)$

For $R \in \mathcal{R}(A)$, we note R_1 the binary relation on A defined by (for all $x,\ y \in A$):

$$x\, R_1\, y \Leftrightarrow \Phi_R(x, y) > \Phi_R(y, x).$$

Since for $x,\ y \in A$ and $m,\ n \in \mathbb{N}^*$, we have $\Phi_R(mx, ny) = \frac{m}{n}\Phi_R(x, y)$, R_1 is still a positive homothetic order. For $R \in \mathcal{R}(A)$, the indifference relation $S_1 = R_1^\vee \cap (-R_1)$ associated with R_1, is given by (for $x,\ y \in A$)

$$x\, S_1\, y \Leftrightarrow \Phi_R(x, y) = \Phi_R(y, x).$$

In particular, S_1 is reflexive. Moreover, we have the

Proposition 11.1. *Let $R \in \mathcal{R}_\bullet(A)$. Then R_1 is a generalized homothetic weak order.*

Proof. Let (u_1, u_2) be a representation of R. We must define a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}^\natural_{>0}$ such that for all $x,\ y \in A$, we have $x\, R_1\, y \Leftrightarrow u(x) > u(y)$.

If $R \in \{R_\emptyset, R_\infty\}$, then $R_1 = R_\emptyset$, and the constant functions $u = 0^+$ or $u = \infty$ can be chosen. We can thus suppose that $R \notin \{R_\emptyset, R_\infty\}$. By (6.4) and (6.5), we can also suppose that:

- if $R \in \mathcal{R}^{\emptyset,\infty}_\bullet(A)$, then (u_1, u_2) is the representation of R in $\{0^+, \infty\}$;
- if $R \in \mathcal{R}_\bullet(A) \setminus \mathcal{R}^{\emptyset,\infty}_\bullet(A)$, then $u_1(\mathcal{A}_1^R) \subset \mathbb{A}$ and $u_2(A) \subset \mathbb{A}$.

Since $R \notin \{R_\emptyset, R_\infty\}$, we have $u_1 \neq 0^+$ and $u_2 \neq \infty$. And we also have:

- if $u_2 = 0^+$, then $R \in \mathcal{R}^{\emptyset,\infty}_\bullet(A)$ and $u_1(A) = \{0^+, \infty\}$;
- if $u_1 = \infty$, then $R \in \mathcal{R}^{\emptyset,\infty}_\bullet(A)$ and $u_2(A) = \{0^+, \infty\}$.

If $u_2 = 0^+$, then for $x,\ y \in A$, we have

$$\Phi_R(x, y) = \begin{cases} \infty & \text{if } u_1(x) = \infty \\ 0^+ & \text{if } u_1(x) = 0^+; \end{cases}$$

in particular, R_1 is a generalized homothetic weak order represented by the function $u = u_1$. If now $u_1 = \infty$, then for $x,\ y \in A$, we have

$$\Phi_R(x, y) = \begin{cases} \infty & \text{if } u_2(y) = 0^+ \\ 0^+ & \text{if } u_2(y) = \infty; \end{cases}$$

in particular, R_1 is a generalized homothetic weak order represented by the function $u = u_2^\vee$, defined by $u(x) = u_2(x)^\vee$. We can then suppose that $u_2 \neq 0^+$ and $u_1 \neq \infty$.

By ($_5$S), for $(x, y) \in \Phi_R^{-1}(\{0^+, \infty\})$, we have $\Phi_R(x, A_2^R) = \Phi_R(x, y)$ or $\Phi_R(A_1^R, y) = \Phi_R(x, y)$. And for $x,\ y \in A$, by the hypothesis above, we have

$$\Phi_R(x, A_2^R) = \infty \Leftrightarrow u_1(x) = \infty,$$
$$\Phi_R(x, A_2^R) = 0^+ \Leftrightarrow u_1(x) = 0^+,$$
$$\Phi_R(A_1^R, y) = \infty \Leftrightarrow u_2(y) = 0^+,$$
$$\Phi_R(A_1^R, y) = 0^+ \Leftrightarrow u_2(y) = \infty.$$

Recall that $A_{1,2}^R = \{x \in A : u_1(x) \notin \{0^+, \infty\}\}$ and $A_{2,1}^R = \{x \in A : u_2(x) \notin \{0^+, \infty\}\}$. Hence for $x \in A \setminus (A_{1,2}^R \cap A_{2,1}^R) = (A \setminus A_{1,2}^R) \cup (A \setminus A_{2,1}^R)$, there exists a $i \in \{1, 2\}$ such that $u_i(x) \in \{0^+, \infty\}$, and we can let

$$u(x) = \begin{cases} \infty & \text{if } u_1(x) = \infty \text{ or } u_2(x) = 0^+ \\ 0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty. \end{cases}$$

From what precedes, the element $u(x) \in \{0^+, \infty\}$ is well-defined. Besides, for $x \in A_{1,2}^R \cap A_{2,1}^R$, since $u_1(x) \in \mathbb{R}^\natural_{>0} \setminus \{0^+, \infty\}$ and $u_2(x) \in \mathbb{R}_{>0}$, we can let

$$v(x) = u_2(x) \cdot u_1(x) \in \mathbb{R}^\natural_{>0} \setminus \{0^+, \infty\}$$

and

$$u(x) = \begin{cases} v(x)^{1/2} & \text{if } v(x) \in \mathbb{R}_{>0} \\ (r^{1/2})^+ & \text{if } v(x) = r^+. \end{cases}$$

The function $u : A \to \mathbb{R}^\natural_{>0}$ thereby defined, is a morphism of $\mathbb{N}^*$-sets. And for $x \in A$, we have

$$x \in \mathcal{A}_1^R \Leftrightarrow u(x) \in \mathbb{A}.$$

We must check that for all $x,\ y \in A$, we have

$$\Phi_R(x,y) > \Phi_R(y,x) \Leftrightarrow u(x) > u(y).$$

Take $x,\ y \in A$. If $x,\ y \in A^R_{1,2} \cap A^R_{2,1}$, we have

$$\begin{aligned}\Phi_R(x,y) > \Phi_R(y,x) &\Leftrightarrow \Phi_>(u_1(x), u_2(y)) > \Phi_>(u_1(y), u_2(x))\\ &\Leftrightarrow u_2(y)^{-1}\cdot u_1(x) > u_2(x)^{-1}\cdot u_1(y)\\ &\Leftrightarrow v(x) > v(y)\\ &\Leftrightarrow u(x) > u(y).\end{aligned}$$

If $(x,y) \in \Phi_R^{-1}(\{0^+,\infty\})$ and $(y,x) \in \Phi_R^{-1}(\mathbb{B} \smallsetminus \{0^+,\infty\})$, we have

$$\begin{aligned}\Phi_R(x,y) > \Phi_R(y,x) &\Leftrightarrow \Phi_R(x,A) = \infty \text{ or } \Phi_R(A,y) = \infty\\ &\Leftrightarrow u(x) = \infty \text{ or } u(y) = 0^+.\\ &\Leftrightarrow u(x) > u(y).\end{aligned}$$

If $(x,y) \in \Phi_R^{-1}(\mathbb{B} \smallsetminus \{0^+,\infty\})$ and $(y,x) \in \Phi_R^{-1}(\{0^+,\infty\})$, we have

$$\begin{aligned}\Phi_R(x,y) > \Phi_R(y,x) &\Leftrightarrow \Phi_R(y,A) = 0^+ \text{ or } \Phi_R(A,x) = 0^+\\ &\Leftrightarrow u(y) = 0^+ \text{ or } u(x) = \infty.\\ &\Leftrightarrow u(x) > u(y).\end{aligned}$$

Finally, if $(x,y),\ (y,x) \in \Phi_R(\{0^+,\infty\})$, we have

$$\begin{aligned}\Phi_R(x,y) > \Phi_R(y,x) &\Leftrightarrow \Phi_R(x,y) = \infty \text{ and } \Phi_R(y,x) = 0^+\\ &\Leftrightarrow u(x) = \infty \text{ and } u(y) = 0^+.\\ &\Leftrightarrow u(x) > u(y).\end{aligned}$$

This ends the proof of the proposition: R_1 is a generalized homothetic weak order, represented by u. □

To formulate the following results, it is convenient to write:

$$\begin{aligned}&\infty\cdot r = \infty \quad (r \in \mathbb{R}^\natural_{>0}),\\ &0^+\cdot r = 0^+ \quad (r \in \mathbb{R}^\natural_{>0}),\\ &\infty^{-1} = 0^+,\\ &(0^+)^{-1} = \infty.\end{aligned}$$

Beware: we have $\infty\cdot 0^+ = \infty$ and $0^+\cdot\infty = 0^+$.

Corollary 11.2. *Let $R \in \mathcal{R}_\bullet(A)$, and let (u_1,u_2) be a representation of R such that:*

- *if $R = R_\emptyset$, then $(u_1,u_2) = (0^+,\infty)$;*
- *if $R \in \mathcal{R}^{\emptyset,\infty}_\bullet(A)$, then (u_1,u_2) is the representation of R in $\{0^+,\infty\}$;*
- *If $R \in \mathcal{R}_\bullet(A) \smallsetminus \mathcal{R}^{\emptyset,\infty}_\bullet(A)$, then $u_1(\mathcal{A}^R_1) \subset \mathbb{A}$ and $u_2(A) \subset \mathbb{A}$.*

Thus, the function $u : A \to \mathbb{R}^\natural_{>0}$ defined by

$$u(x) = \begin{cases}\infty & \text{if } u_1(x) = \infty \text{ and } u_2(x) \neq \infty\\ \infty & \text{if } u_1(x) \neq 0^+ \text{ and } u_2(x) = 0^+\\ 0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty\\ r^{1/2} & \text{if } u_2(x)\cdot u_1(x) = r \in \mathbb{R}_{>0}\\ (r^{1/2})^+ & \text{if } u_2(x)\cdot u_1(x) = r^+ \in \mathbb{R}^\natural_{>0} \smallsetminus \mathbb{A}\end{cases}$$

is a representation of R_1. And letting $\gamma, \gamma^- : A \to \mathbb{A}$ be the functions defined by

$$\gamma(x) = \begin{cases}\infty & \text{if } u_1(x) = \infty \text{ and } u_2(x) \neq \infty\\ \infty & \text{if } u_1(x) \neq 0^+ \text{ and } u_2(x) = 0^+\\ 0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty\\ [u_2(x)^{-1}\tilde{u}_1(x)]^{1/2} & \text{otherwise}\end{cases}$$

and

$$\gamma^-(x) = \gamma(x)^{-1},$$

we have $(\gamma\cdot u, \gamma^-\cdot\tilde{u}) = (u_1,u_2)$.

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Proof. If $R = R_\emptyset$, then $u = 0^+$, $\gamma = 0^+$ and $(\gamma \cdot u, \gamma^- \cdot \tilde{u}) = (0^+, \infty) = (u_1, u_2)$. If $R = R_\infty$, then $u = \infty$, $\gamma = \infty$ and $(\gamma \cdot u, \gamma^- \cdot \tilde{u}) = (\infty, 0^+) = (u_1, u_2)$. Note that in both cases, u is a representation of $R_1 = R_\emptyset$. If now $R \notin \{R_\emptyset, R_\infty\}$, then u is the representation of R_1 built in the proof of (11.1); and we verify that $(\gamma \cdot u, \gamma^- \cdot \tilde{u}) = (u_1, u_2)$. □

Let $A/\mathbb{N}^*$ be the quotient-set of A by the equivalence relation $\sim_{\mathbb{N}^*}$ on A defined by

$x \sim_{\mathbb{N}^*} y \Leftrightarrow$ there exist $m, n \in \mathbb{N}^*$ such that $mx = ny$.

Corollary 11.3. *Let $R \in \mathcal{R}_\bullet(A)$. There exist a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}^\natural_{>0}$ and a map $\gamma : A/\mathbb{N}^* \to \mathbb{A}$ such that (for all $x, y \in A$)*

(i) $u(\mathcal{A}_1^R) \subset \mathbb{A}$,

(ii) $\gamma^{-1}(\infty) = u^{-1}(\infty)$,

(iii) $\gamma^{-1}(0^+) = u^{-1}(0^+)$,

(iv) $x\,R\,y \Leftrightarrow \gamma(x) \cdot u(x) > \gamma(y)^{-1} \cdot \tilde{u}(y)$.

Moreover, up to multiplication by an element of $\mathbb{R}_{>0}$, the pair (u, γ) is unique: if (u', γ') is another pair of maps like above and satisfying the conditions (i), (ii), (iii), (iv), *then there exists a $\lambda \in \mathbb{R}_{>0}$ such that $(u', \gamma') = (\lambda \cdot u, \gamma)$.*

Proof. The existence of the pair (u, γ) results from the Corollary 11.2; note that by construction, $u : A \to \mathbb{R}^\natural_{>0}$ is a morphism of $\mathbb{N}^*$-sets, and $\gamma : A \to \mathbb{A}$ factorizes through $A/\mathbb{N}^*$. The uniqueness of the pair (u, γ) is a consequence of the uniqueness property in Lemmas 6.4 and 6.5. □

Corollary 11.4. *Let $R \in \mathcal{R}_\bullet(A)$, and let $u : A \to \mathbb{R}^\natural_{>0}$ and $\gamma : A/\mathbb{N}^* \to \mathbb{A}$ be a morphism of $\mathbb{N}^*$-sets and a map, satisfying the conditions* (i), (ii), (iii), (iv) *of (11.3). Then u represents R_1.*

Remark 11.5. For $R \in \mathcal{R}(A) \setminus \mathcal{R}_\bullet(A)$, the relation R_1 is not always a generalized homothetic weak order. We can for instance verify that the relation R of the Remark 9.5 satisfies $R_1 = R$. ⋆

Remark 11.6. For $R \in \mathcal{R}(A)$ and $n \in \mathbb{N}^*$, we define by induction an order $R_{n+1} \in \mathcal{R}(A)$: we put $R_{n+1} = (R_n)_1$. For all $R \in \mathcal{R}(A)$ and all $n \in \mathbb{N}^*$, one can verify that $R_n = R_1$. ⋆

12. Comments

The Corollary 11.3 is a generalization of [13]. Indeed, in [13] we have obtained the same result but only for homothetic interval orders on A. Representing a relation $R \in \mathcal{R}_\bullet(A)$ by a pair (u, γ) as in (11.3) rather than by a pair (u_1, u_2) like in (6.4) and (6.5), has the advantage of showing the underlying generalized homothetic weak order R_1 (represented by u). We can then "see" R as a deformation of R_1, the deformation being represented by the twisting factor $\gamma : A \to \mathbb{A}$. This naturally leads to group in a single family the relations $R \in \mathcal{R}_\bullet(A)$ having the same underlying generalized homothetic weak order R_1.

The introduction of the set $\mathbb{R}^\natural_{>0}$ is not merely an ad hoc construction to treat the abandon of the super-Archimedean property. Recall that $\mathbb{R}^\natural_{>0}$ is the set of intervals of $\mathbb{R}_{>0}$ of the form $[r, +\infty[$ and $]r, \infty[$, to which the empty interval is added. The name itself of "interval order" naturally leads to the following question: why limiting oneself to relations that can be represented by closed intervals, and not consider the relations that can be by intervals which are closed *or* open. The set $\mathbb{R}^\natural_{>0}$ is a response to this question. Another response is given by the following variant of Lemma 6.5:

Lemma 12.1. *Let $R \in \mathcal{R}_\bullet(A) \setminus \mathcal{R}_\bullet^{\emptyset,\infty}(A)$. There exist two morphisms of $\mathbb{N}^*$-sets $v_1, v_2 : A \to \mathbb{A}$ such that for all $x, y \in A$, we have*

$$x\,R\,y \Leftrightarrow \begin{cases} v_1(x) > v_2(y) & \text{if } (x, y) \in \mathcal{A}^R \\ v_1(x) \geq v_2(y) & \text{if } (x, y) \in \mathcal{B}^R. \end{cases}$$

Moreover, up to multiplication by an element of $\mathbb{R}_{>0}$, the pair (v_1, v_2) is unique.

Proof. By (6.5), there exists a representation (u_1, u_2) of R such that $u_1(\mathcal{A}_1^R) \subset \mathbb{A}$ and $u_2(A) \subset \mathbb{A}$. Consider the projection $\mathbb{R}^\natural_{>0} \to \mathbb{A}$, $r \to \tilde{r}$ defined in the proof of (9.3). And for any function $u : A \to \mathbb{R}^\natural_{>0}$, note $\tilde{u} : A \to \mathbb{A}$ the function defined by $\tilde{u}(x) = \widetilde{u(x)}$. Then, the pair $(v_1, v_2) = (\tilde{u}_1, u_2)$ satisfies the conditions of the lemma. And the uniqueness property of (v_1, v_2) results from the uniqueness property of (u_1, u_2). □

In our opinion, the answer (6.5) is preferable to the answer (12.1). Indeed, in (12.1), we must first choose whether a triplet (x, R, y) is or is not super-Archimedean *before* being able to decide whether $x\,R\,y$ or $x\,(-R)\,y$ with the pair of functions (v_1, v_2). On the other hand, in (6.5), the fact that a triplet (x, R, y) is or is not super-Archimedean is deduced *a posteriori* from the representation (u_1, u_2); i.e. the pair of values $(u_1(x), u_2(y)) \in \mathbb{R}^\natural_{>0} \times \mathbb{R}^\natural_{>0}$ allows not only deciding if $x\,R\,y$ or $x\,(-R)\,y$, but also deciding whether (x, R, y) is super-Archimedean or not.

The study of positive homothetic orders on A which are not generalized homothetic biorders will be the focus of a further work.

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